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Unitarity, crossing symmetry and duality in the scattering of $\mathcal{N}=1$ susy matter Chern-Simons theories

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Abstract: We study the most general renormalizable $\mathcal{N}=1$ U(N) Chern-Simons gauge theory coupled to a single (generically massive) fundamental matter multiplet. At leading order in the 't Hooft large N limit we present computations and conjectures for the 2×2 S matrix in these theories; our results apply at all orders in the 't Hooft coupling and the matter self interaction. Our S matrices are in perfect agreement with the recently conjectured strong weak coupling self duality of this class of theories. The consistency of our results with unitarity requires a modification of the usual rules of crossing symmetry in precisely the manner anticipated in arXiv:1404.6373, lending substantial support to the conjectures of that paper. In a certain range of coupling constants our S matrices have a pole whose mass vanishes on a self dual codimension one surface in the space of couplings.

Keywords: Supersymmetric gauge theory, Duality in Gauge Field Theories, Chern-Simons Theories, 1/N Expansion

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1 Introduction

Non-Abelian U(N) gauge theories in three spacetime dimensions are dynamically rich. At low energies parity preserving gauge self interactions are generically governed by the Yang-Mills action

$$\frac{1}{g_{\rm YM}^2}\int d^3x \ {\rm Tr}\ F_{\mu\nu}^2 \,. \tag{1.1}$$

As $g_{\rm YM}^2$ has the dimensions of mass, gluons are strongly coupled in the IR. In the absence of parity invariance the gauge field Lagrangian generically includes an additional Chern-Simons term and schematically takes the form

$$\frac{i\kappa}{4\pi}\int {\rm Tr}\left(AdA+\frac{2}{3}A^3\right)-\frac{1}{4g_{\rm YM}^2}\int d^3x \ {\rm Tr}\ F_{\mu\nu}^2 \,. \tag{1.2}$$

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The Lagrangian (1.2) describes a system of *massive* gluons; with mass $m \propto \kappa g_{\rm YM}^2$. At energies much lower than $g_{\rm YM}^2$ (1.2) has no local degrees of freedom. The effective low energy dynamics is topological, and is governed by the action (1.2) with the Yang-Mills term set to zero. This so called pure Chern-Simons theory was solved over twenty five years ago by Witten [1]; his beautiful and nontrivial exact solution has had several applications in the study of two dimensional conformal field theories and the mathematical study of knots on three manifolds.

Let us now add matter fields with standard, minimally coupled kinetic terms, (in any representation of the gauge group) to (1.2). The resulting low energy dynamics is particularly simple in the limit in which all matter masses are parametrically smaller than $g_{\rm YM}^2$. In order to focus on this regime we take the limit $g_{\rm YM}^2 \to \infty$ with masses of matter fields held fixed. In this limit the Yang-Mills term in (1.2) can be ignored and we obtain a Chern-Simons self coupled gauge theory minimally coupled to matter fields. While the gauge fields are non propagating, they mediate nonlocal interactions between matter fields.

In order to gain intuition for these interactions it is useful to first consider the special case $N = 1$, i.e. the case of an Abelian gauge theory interacting with a unit charge scalar field. The gauge equation of motion

$$\kappa \varepsilon^{\mu\nu\rho} F_{\nu\rho} = 2\pi J^\mu \tag{1.3}$$

ensures that each matter particle traps $\frac{1}{\kappa}$ units of flux (where $i \int F = 2\pi$ is defined as a single unit of flux). It follows as a consequence of the Aharonov-Bohm effect that exchange of two unit charge particles results in a phase $\frac{\pi}{\kappa}$; in other words the Chern-Simons interactions turns the scalars into anyons with anyonic phase $\pi\nu = \frac{\pi}{\kappa}$.

The interactions induced between matter particles by the exchange of non-abelian Chern-Simons gauge bosons are similar with one additional twist. In close analogy with the discussion of the previous paragraph, the exchange of two scalar matter quanta in representations R_1 and R_2 of $\mathrm{U}(N)$ results in the phase $\frac{\pi T_{R_1}.T_{R_2}}{\kappa}$ where T_R is the generator of $\mathrm{U}(N)$ in the representation R. The new element in the non-abelian theory is that the phase obtained upon interchanging two particles is an operator (in $\mathrm{U}(N)$ representation space) rather than a number. The eigenvalues of this operator are given by

$$\nu'_R = \frac{c_2(R_1) + c_2(R_2) - c_2(R')}{2\kappa} \tag{1.4}$$

where $c_2(R)$ is the quadratic Casimir of the representation R and R' runs over the finite set of representations that appear in the Clebsh-Gordon decomposition of the tensor product of R_1 and R_2. In other words the interactions mediated by non-abelian Chern-Simons coupled gauge fields turns matter particles into non-abelian anyons.

In some ways anyons are qualitatively different from either bosons or fermions. For example anyons (with fixed anyonic phases) are never free: there is no limit in which the multi particle anyonic Hilbert space can be regarded as a 'Fock space' of a single particle state space. Thus while matter Chern-Simons theories are regular relativistic quantum field theories from a formal viewpoint, it seems possible that they will display dynamical features never before encountered in the study of quantum field theories. This possibility provides one motivation for the intensive study of these theories.

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Over the last few years matter Chern-Simons theories have been intensively studied in two different contexts. The $\mathcal{N}=6$ supersymmetric ABJ and ABJM theories [2, 3] have been exhaustively studied from the viewpoint of the AdS/CFT correspondence [4, 5]. Several other supersymmetric Chern-Simons theories with $\mathcal{N}\geq 2$ supersymmetry have also been intensively studied, sometimes motivated by brane constructions in string theory. The technique of supersymmetric localization has been used to perform exact computations of several supersymmetric quantities [6–11] (indices, supersymmetric Wilson loops, three sphere partition functions). These studies have led, in particular, to the conjecture and detailed check for 'Seiberg like' Giveon-Kutasov dualities between Chern-Simons matter theories with $\mathcal{N}\geq 2$ supersymmetry [12, 13]. Most of these impressive studies have, however, focused on observables[1] that are not directly sensitive to the anyonic nature of of the underlying excitations and have exhibited no qualitative surprises.

Qualitative surprises arising from the effectively anyonic nature of the matter particles seem most likely to arise in observables built out of the matter fields themselves rather than gauge invariant composites of these fields. There exists a well defined gauge invariant observable of this sort; the S matrix of the matter fields. While this quantity has been somewhat studied for highly supersymmetric Chern-Simons theories, the results currently available (see e.g. [14–20]) have all be obtained in perturbation theory. Methods based on supersymmetry have not yet proved powerful enough to obtain results for S matrices at all orders in the coupling constant, even for the maximally supersymmetric ABJ theory. For a very special class of matter Chern-Simons theories, however, it has recently been demonstrated that large N techniques are powerful enough to compute S matrices at all orders in a 't Hooft coupling constant, as we now pause to review.

Consider large N Chern-Simons coupled to a finite number of matter fields in the fundamental representation of $\mathrm{U}(N)$.[2] It was realized in [21] that the usual large N techniques are roughly as effective in these theories as in vector models even in the absence of supersymmetry (see [22–43] for related works). In particular large N techniques have recently been used in [36] to compute the $2\to 2$ S matrices of the matter particles in purely bosonic/fermionic fundamental matter theories coupled to a Chern-Simons gauge field.

Before reviewing the results of [36] let us pause to work out the effective anyonic phases for two particle systems of quanta in the fundamental/ antifundamental representations at large N.[3] Following [36] we refer to any matter quantum that transforms in the

[1]These observables include partition functions, indices, Wilson lines and correlation functions of local gauge invariant operators. Note that gauge invariant operators do not pick up anyonic phases when they go around each other precisely because they are gauge singlets.

[2]These theories were initially studied because of their conjectured dual description in terms of Vasiliev equations of higher spin gravity.

[3]The application of large N techniques to these theories has led to conjectures for strong weak coupling dualities between classes of these theories. The simplest such duality relates a Chern-Simons theory coupled to a single fundamental bosonic multiplet to another Chern-Simons theory coupled to a single fermionic multiplet. This duality was first clearly conjectured in [26], building on the results of [23, 24], and following up on an earlier suggestion in [21]. The discovery of a three dimensional Bose-Fermi duality was the first major qualitative surprise in the study of Chern-Simons matter theories, and is intimately connected with the effectively anyonic nature of the matter excitations, as explained, for instance, in [36].

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(anti)fundamental of U(N) a(n) (anti)particle. A two particle system can couple into two representations R' (see (1.4)); the symmetric representation (two boxes in the first row of the Young tableaux) and the antisymmetric representation (two boxes in the first column of the Young tableaux). It is easily verified that the anyonic phase $\nu_{R'}$ (see (1.4)) is of order $\frac{1}{N}$ (and so negligible in the large N limit) for both choices of R'. On the other hand a particle-antiparticle system can couple into R' which is either the adjoint of the singlet. $\nu_{R'}$ once again vanishes in the large N limit when R' is the adjoint. However when R' is the singlet representation it turns out that $\nu_{sing} = \frac{N}{\kappa} = \lambda$ and so is of order unity in the large N limit. In summary two particle systems are always non anyonic in the large N limit of these special theories. Particle-antiparticle systems are also non anyonic in the adjoint channel. However they are effectively anyonic — with an interesting finite anyonic phase — in the singlet channel. See [36] for more details. This preparation makes clear that qualitative surprises related to anyonic physics in the two quantum scattering in these theories might occur only in particle-antiparticle scattering in the singlet sector.

The authors of [36] used large N techniques to explicitly evaluate the S matrices in all three non-anyonic channels in the theories they studied (see below for more details of this process). They also used a mix of consistency checks and physical arguments involving crossing symmetry to conjecture a formula for the particle-antiparticle S matrix in the singlet channel. The conjecture of [36] for the S matrix in the singlet channel has two unexpected novelties related to the anyonic nature of the two particle state

1. The singlet S matrix in both the bosonic and fermion theories has a contact term localized on forward scattering. In particular the S matrix is not an analytic function of momenta.

2. The analytic part of the singlet S matrix is given by the analytic continuation of the S matrix in any of the other three channels $\times\ \frac{\sin \pi\lambda}{\pi\lambda}$. In other words the usual rules of crossing symmetry to the anyonic channel are modified by a factor determined by the anyonic phase.

The modification of the usual rules of analyticity and crossing symmetry in the anyonic channel of 2×2 scattering was a major surprise of the analysis of [36]. The authors of [36] offered physical explanations — involving the anyonic nature of scattering in the singlet channel for both these unusual features of the S matrix. The simple (though non rigorous) explanations proposed in [36] are universal in nature; they should apply equally well to all large N Chern-Simons theories coupled to fundamental matter, and not just the particular theories studied in [36]. This fact suggests a simple strategy for testing the conjectures of [36] which we employ in this paper. We simply redo the S matrix computations of [36] in a different class of Chern-Simons theory coupled to fundamental matter and check that the conjectures of [36] — unmodified in all details — indeed continue to yield sensible results (i.e. results that pass all necessary consistency checks) in the new system. We now describe the system we study and the nature of our results in much more detail.

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The theories we study are the most general power counting renormalizable $\mathcal{N} = 1$ U(N) gauge theories coupled to a single fundamental multiplet (see (2.1) below). In order to study scattering in these theories we imitate the strategy of [36]. The authors of [36] worked in lightcone gauge; in this paper we work in a supersymmetric generalization of lightcone gauge (3.1). In this gauge (which preserves manifest offshell supersymmetry) the gauge self interaction term vanishes. This fact — together with planarity at large N — allows us to find a manifestly supersymmetric Schwinger-Dyson equation for the exact propagator of the matter supermultiplet. This equation turns out to be easy to solve; the solution gives simple exact expression for the all orders propagator for the matter supermultiplet (see subsection 3.3).

With the exact propagator in hand, we then proceed to write down an exact Schwinger-Dyson equation for the offshell four point function of the matter supermultiplet. The resultant integral equation is quite complicated; as in [36] we have been able to solve this equation only in a restricted kinematic range ($q_\pm = 0$ in the notation of figure 4). In this kinematic regime, however, we have been able to find a completely explicit (if somewhat complicated) solution of the resulting equation (see subsections 3.5–3.6).

In order to evaluate the S matrices we then proceed to take the onshell limit of our explicit offshell results. As explained in detail in [36], the 3 vector q^μ has the interpretation of momentum transfer for both channels of particle-particle scattering and also for particle antiparticle scattering in the adjoint channel. In these channels the fact that we know the offshell four point amplitudes only when $q_\pm = 0$ forces us to study scattering in a particular Lorentz frame; any frame in which momentum transfer happens along the spatial q^3 direction. In any such frame we obtain explicit results for all 2×2 scattering matrices in these three channels. The results are then covariantized to formulae that apply to any frame. Following this method we have obtained explicit results for the S matrices in these three channels. Our results are presented in detail in subsections 3.7–3.11. As we explain in detail below, our explicit results have exactly the same interplay with the proposed strong weak coupling self duality of the set of $\mathcal{N} = 1$ Chern-Simons fundamental matter theories (see subsection 2.2) as that described in [36]; duality maps particle-particle S matrices in the symmetric and antisymmetric channels to each other, while it maps the particle-antiparticle S matrix in the adjoint channel to itself.

As in [36] our explicit offshell results do not permit a direct computation of the S matrix for particle-antiparticle scattering in the singlet channel. This is because the three vector q^μ is the center of mass momentum for this scattering process and so must be timelike, which is impossible if $q^\pm = 0$. Our explicit results for the S matrices in the other channels, together with the conjectured modified crossing symmetry rules of [36], however, yield a conjectured formula for the S matrix in this channel.

In section 4 we subject our conjecture for the particle-antiparticle S matrix to a very stringent consistency check; we verify that it obeys the nonlinear unitarity equation (2.61).[4] From the purely algebraic point of view the fact that our complicated S matrices are unitary appears to be a minor miracle- one that certainly fails very badly for the S matrix obtained using the usual rules of crossing symmetry. We view this result as very strong evi-

[4] At large N this equation may be shown to close on 2×2 scattering.

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dence for the correctness of our formula, and indirectly for the modified crossing symmetry rules of [36].

Our proposed formula for particle-antiparticle scattering in the singlet channel has an interesting analytic structure. As a function of s (at fixed t) our S matrix has the expected two particle cut starting at $s = 4m^2$. In a certain range of interaction parameters it also has poles at smaller (though always positive) values of s. These poles represent bound states; when they exist these bound states must be absolutely stable even at large but finite N, simply because they are the lightest singlet sector states (baring the vacuum) in the theory; recall that our theory has no gluons. Quite remarkably it turns out that the mass of this bound state supermultiplet vanishes at $w = w_c(\lambda)$ where w is the superpotential interaction parameter of our theory (see (2.1)) and $w_c(\lambda)$ is the simple function listed in (5.11). In other words a one parameter tuning of the superpotential is sufficient to produce massless bound states in a theory of massive 'quarks'; we find this result quite remarkable. Scaling w to w_c permits a parametric separation between the mass of this bound state and all other states in the theory. In this limit there must exist a decoupled QFT description of the dynamics of these light states even at large but finite N; it seems likely to us that this dynamics is governed by a $\mathcal{N} = 1$ Wilson-Fisher fixed point. We leave the detailed investigation of this suggestion to future work.

The S matrices computed and conjectured in this paper turn out to simplify dramatically at $w = 1$, at which point the system (2.1) turns out to enjoy an enhanced $\mathcal{N} = 2$ supersymmetry. In the three non-anyonic channels our S matrix reduces simply to its tree level counterpart at $w = 1$. It follows, in other words, that the S matrix is not renormalized as a function of λ in these channels. This result illustrates the conflict between naive crossing symmetry and unitarity in a simple setting. Naive crossing symmetry would yield a singlet channel S matrix that is also tree level exact. However tree level S matrices by themselves can never obey the unitarity equations (they do not have the singularities needed to satisfy the Cutkosky's rules obtained by gluing them together). The resolution to this paradox appears simply to be that the naive crossing symmetry rules are wrong in the current context. Applying the conjectured crossing symmetry rules of [36] we find a singlet channel S matrix that continues to be very simple, but is not tree level exact, and in fact satisfies the unitarity equation.

In this paper we have limited our attention to the study of $\mathcal{N} = 1$ theories with a single fundamental matter multiplet. Were we to extend our analysis to theories with two multiplets we would encounter, in particular, the $\mathcal{N} = 3$ theory. Extending to the study of a theory with four multiplets (and allowing for the the gauging of a U(1) global symmetry) would allow us to study the $\mathcal{N} = 6$ $\mathrm{U}(N) \times \mathrm{U}(1)$ ABJ theory. We believe it would not be difficult to adapt the methods of this paper to find explicit all orders results for the S matrices of all these theories at leading order in large N. We expect to find scattering matrices that are unitary precisely because they transform under crossing symmetry in the unusual manner conjectured in [36]. It would be particularly interesting to find explicit results for the $\mathcal{N} = 6$ theory in order to facilitate a detailed comparison with the perturbative computations of S matrices in ABJM theory [14–20], which appear to report results that are crossing symmetric but (at least naively) conflict with unitarity.

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2 Review of background material

2.1 Renormalizable $\mathcal{N}=1$ theories with a single fundamental multiplet

In this paper we study 2×2 scattering in the most general renormalizable $\mathcal{N}=1$ supersymmetric U(N) Chern-Simons theory coupled to a single fundamental matter multiplet. Our theory is defined in superspace by the Euclidean action [44, 45]

$$\begin{aligned}\mathcal{S}_{\mathcal{N}=1} = -\int d^3x d^2\theta \Bigg[&\frac{\kappa}{2\pi} Tr\left(-\frac{1}{4}D_\alpha\Gamma^\beta D_\beta\Gamma^\alpha - \frac{1}{6}D^\alpha\Gamma^\beta\{\Gamma_\alpha,\Gamma_\beta\} - \frac{1}{24}\{\Gamma^\alpha,\Gamma^\beta\}\{\Gamma_\alpha,\Gamma_\beta\}\right)\\ &-\frac{1}{2}(D^\alpha\bar{\Phi}+i\bar{\Phi}\Gamma^\alpha)(D_\alpha\Phi - i\Gamma_\alpha\Phi) + m_0\bar{\Phi}\Phi + \frac{\pi w}{\kappa}(\bar{\Phi}\Phi)^2\Bigg]. \end{aligned} \tag{2.1}$$

The integration in (2.1) is over the three Euclidean spatial coordinates and the two anticommuting spinorial coordinates θ^α (the SO(3) spinorial indices α range over two allowed values $\pm$). The fields Φ and Γ^α in (2.1) are, respectively, complex and real superfields.[5] They may be expanded in components as

$$\begin{aligned}\Phi &= \phi + \theta\psi - \theta^2 F\,,\\ \bar{\Phi} &= \bar{\phi} + \theta\bar{\psi} - \theta^2\bar{F}\,,\\ \Gamma^\alpha &= \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta^{\ \alpha} - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta)\,,\end{aligned} \tag{2.2}$$

where Γ_α is an $N\times N$ matrix in color space, while Φ is an N dimensional column.

The superderivative D_α in (2.1) is defined by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\alpha\beta}\,, D^\alpha = C^{\alpha\beta}D_\beta\,, \tag{2.3}$$

where $C^{\alpha\beta}$ is the charge conjugation matrix. See appendix A.1 for notations and conventions.

The theories (2.1) are characterized by one dimensionless coupling constant w, a dimensionful mass scale m_0, and two integers N (the rank of the gauge group U(N)) and κ, the level of the Chern-Simons theory.[6] In the large N limit of interest to us in this paper, the 't Hooft coupling $\lambda = \frac{N}{\kappa}$ is a second effectively continuous dimensionless parameter.

The action (2.1) enjoys invariance under the super gauge transformations

$$\begin{aligned}\delta\Phi &= iK\Phi\,,\\ \delta\bar{\Phi} &= -i\bar{\Phi}K\,,\\ \delta\Gamma_\alpha &= D_\alpha K + [\Gamma_\alpha, K]\,,\end{aligned} \tag{2.4}$$

where K is a real superfield (it is an $N\times N$ matrix in color space).

(2.1) is manifestly invariant under the two supersymmetry transformations generated by the supercharges Q_α

$$Q_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right) \tag{2.5}$$

[5] See appendix A.2 for our conventions for superspace.

[6] The precise definition of κ is defined as follows. Let k denote the level of the WZW theory related to Chern-Simons theory after all fermions have been integrated out. κ is the related to k by $\kappa = k + \mathrm{sgn}(k)N$.

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that act on Φ and Γ_α as

$$\begin{aligned}\delta_\alpha \Phi &= Q_\alpha \Phi\,,\\ \delta_\alpha \Gamma_\beta &= Q_\alpha \Gamma_\beta\,.\end{aligned} \tag{2.6}$$

The differential operators Q_α and D_α obey the algebra

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= 2i\partial_{\alpha\beta}\,,\\ \{D_\alpha, D_\beta\} &= 2i\partial_{\alpha\beta}\,,\\ \{Q_\alpha, D_\beta\} &= 0\,.\end{aligned} \tag{2.7}$$

At the special value $w = 1$, the action (2.1) actually has enhanced supersymmetry; it is $\mathcal{N} = 2$ (four supercharges) supersymmetric.[7]

The physical content of the theory (2.1) is most transparent when the Lagrangian is expanded out in component fields in the so called Wess-Zumino gauge — defined by the requirement

$$B = 0\,, \chi = 0\,. \tag{2.8}$$

Imposing this gauge, integrating over θ and eliminating auxiliary fields we obtain the component field action[8]

$$\begin{aligned}\mathcal{S}_{\mathcal{N}=1} = \int d^3x \Bigg(&-\frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + \mathcal{D}^\mu\bar\phi\mathcal{D}_\mu\phi + m_0^2\bar\phi\phi - \bar\psi(i\not{\mathcal{D}} + m_0)\psi\\ &+ \frac{4\pi w m_0}{\kappa}(\bar\phi\phi)^2 + \frac{4\pi^2 w^2}{\kappa^2}(\bar\phi\phi)^3 - \frac{2\pi}{\kappa}(1+w)(\bar\phi\phi)(\bar\psi\psi) - \frac{2\pi w}{\kappa}(\bar\psi\phi)(\bar\phi\psi)\\ &+ \frac{\pi}{\kappa}(1-w)\big((\bar\phi\psi)(\bar\phi\psi) + (\bar\psi\phi)(\bar\psi\phi)\big)\Bigg)\end{aligned} \tag{2.11}$$

displaying that (2.1) is the action for one fundamental boson and one fundamental fermion coupled to a Chern-Simons gauge field. Supersymmetry sets the masses of the bosonic and fermionic fields equal, and imposes several relations between a priori independent coupling constants.

2.2 Conjectured duality

It has been conjectured [33] that the theory (2.1) enjoys a strong weak coupling self duality. The theory (2.1) with 't Hooft coupling λ and self coupling parameter w is conjectured to be dual to the theory with 't Hooft coupling λ' and self coupling w' where

$$\lambda' = \lambda - \mathrm{Sgn}(\lambda)\,,\ w' = \frac{3-w}{1+w} \quad m_0' = \frac{-2m_0}{1+w}\,. \tag{2.12}$$

[7] This may be confirmed, for instance, by checking that (2.11) at $w = 1$ is identical to the $\mathcal{N} = 2$ superspace Chern-Simons action coupled to a single chiral multiplet in the fundamental representation with no superpotential (see eq. 2.3 of [46]) expanded in components in Wess-Zumino gauge.

[8] Our trace conventions are

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}\,,\ \sum_a (T^a)_i^{\ j}(T^a)_k^{\ l} = \frac{1}{2}\delta_i^{\ l}\delta_k^{\ j}\,. \tag{2.9}$$

The gauge covariant derivatives in (2.11) are

$$\begin{aligned}\mathcal{D}^\mu\bar\phi &= \partial^\mu\bar\phi + i\bar\phi A^\mu\,, & \mathcal{D}_\mu\phi &= \partial_\mu\phi - iA_\mu\phi\,,\\ \not{\mathcal{D}}\bar\psi &= \gamma^\mu(\partial_\mu\bar\psi + i\bar\psi A_\mu)\,, & \not{\mathcal{D}}\psi &= \gamma^\mu(\partial_\mu\psi - iA_\mu\psi).\end{aligned} \tag{2.10}$$

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As we will explain below, the pole mass for the matter multiplet in this theory is given by

$$m = \frac{2m_0}{2 + (-1+w)\lambda \, \mathrm{Sgn(m)}} \,. \tag{2.13}$$

It is easily verified that under duality

$$m' = -m \,. \tag{2.14}$$

The concrete prior evidence for this duality is the perfect matching of S^2 partition functions of the two theories. This match works provided [33]

$$\lambda m(m_0, w) \geq 0 \,, \tag{2.15}$$

Through this paper we will assume that (2.15) is obeyed. Note that the condition (2.15) is preserved by duality (i.e. a theory and its conjectured dual either both obey or both violate (2.15)).

Note that $w = 1$ is a fixed point for the duality map (2.12); this was necessary on physical grounds (recall that our theory has enhanced $\mathcal{N} = 2$ supersymmetry only at $w = 1$). In the special case $w = 1$ and $m_0 = 0$, the duality conjectured in this subsection reduces to the previously studied duality [12] (a variation on Giveon- Kutasov duality [13]). Over the last few years this supersymmetric duality has been subjected to (and has successfully passed) several checks performed with the aid of supersymmetric localization, including the matching of three sphere partition function, superconformal indices and Wilson loops on both sides of the duality [6–11, 30].

2.3 Properties of free solutions of the Dirac equation

In subsequent subsections we will investigate the constraints imposed supersymmetry on the S matrices of the the theory (2.1). Our analysis will make heavy use of the properties of the free solutions to Dirac's equations, which we review in this subsection.

Let u_α and v_α are positive and negative energy solutions to Dirac's equations with mass m. Let $p^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})$. Then u_α and v_α obey

$$\begin{aligned} (\not{p} - m)u(p) &= 0 \,, \\ (\not{p} + m)v(p) &= 0 \,. \end{aligned} \tag{2.16}$$

We choose to normalize these spinors so that

$$\begin{aligned} \bar{u}(\mathbf{p}) \cdot u(\mathbf{p}) &= -2m & \bar{v}(\mathbf{p}) \cdot v(\mathbf{p}) &= 2m \\ u(\mathbf{p})u^*(\mathbf{p}) &= -(\not{p} + m)C & v(\mathbf{p})v^*(\mathbf{p}) &= -(\not{p} - m)C \,. \end{aligned} \tag{2.17}$$

C in (2.17) is the charge conjugation matrix defined to obey the equation

$$C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \,. \tag{2.18}$$

Throughout this paper we use γ matrices that obey the algebra[9]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \,. \tag{2.19}$$

[9] We use the mostly plus convention for $\eta_{\mu\nu}$, the corresponding Euclidean algebra obeys $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$. See appendix A.1 for explicit representations of the γ matrices and charge conjugation matrix C.

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We also choose all three γ^μ matrices to be purely imaginary[10] and to obey

$$(\gamma^\mu)^\dagger = -\eta^{\mu\mu}\gamma^\mu \quad \text{no sum.} \tag{2.20}$$

With these conventions it is easily verified that $C = \gamma^0$ obeys (2.18) and so we choose

$$C = \gamma^0 \,.$$

Using the conventions spelt out above, it is easily verified that $u(\mathbf{p})$ and $v^*(\mathbf{p})$ obey the same equation (i.e. complex conjugation flips the two equations in (2.16)), and have the same normalization. It follows that it is possible to pick the (as yet arbitrary) phases of $u(\mathbf{p})$ and $v(\mathbf{p})$ to ensure that[11]

$$u_\alpha(\mathbf{p}) = -v^*_\alpha(\mathbf{p}), \quad v_\alpha(\mathbf{p}) = -u^*_\alpha(\mathbf{p}) \,. \tag{2.21}$$

We will adopt the choice (2.21) throughout our paper.

Notice that the replacement $m \to -m$ interchanges the equations for u and v. It follows that $u(m) \propto v(-m)$. Atleast with the choice of phase that we adopt in this paper (see below) we find

$$u(m,p) = -v(-m,p), \quad v(m,p) = -u(-m,p) \,. \tag{2.22}$$

To proceed further it is useful to make a particular choice of γ matrices and to adopt a particular choice of phase for u. We choose the γ^μ matrices listed in section A.1 and take $u(\mathbf{p})$ and $v(\mathbf{p})$ to be given by

$$\begin{aligned} u(\mathbf{p}) &= \begin{pmatrix} -\sqrt{p^0 - p^1} \\ \frac{p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix}, \quad \bar{u}(\mathbf{p}) = \begin{pmatrix} \frac{ip^3+m}{\sqrt{p^0-p^1}} & i\sqrt{p^0 - p^1} \end{pmatrix}, \\ v(\mathbf{p}) &= \begin{pmatrix} \sqrt{p^0 - p^1} \\ \frac{-p^3 + im}{\sqrt{p^0 - p^1}} \end{pmatrix}, \quad \bar{v}(\mathbf{p}) = \begin{pmatrix} \frac{-ip^3+m}{\sqrt{p^0-p^1}} & -i\sqrt{p^0 - p^1} \end{pmatrix}, \end{aligned} \tag{2.23}$$

where

$$p^0 = +\sqrt{m^2 + \mathbf{p}^2} \,.$$

Notice that the arguments of the square roots in (2.23) are always positive; the square roots in (2.23) are defined to be positive (i.e. $\sqrt{x^2} = |x|$). It is easily verified that the solutions (2.23) respect (2.22) as promised.

In the rest of this section we discuss an analytic rotation of the spinors to complex (and in particular negative) values of the p^μ (and in particular p^0). This formal construction will prove useful in the study of the transformation properties of the S matrix under crossing symmetry.

---

[10] This is possible in 3 dimensions; recall the unconventional choice of sign in (2.19).

[11] Note that $\bar{u}^\alpha = u^{*\alpha} = C^{\alpha\beta}u^*_\beta$ and not $(u^\alpha)^*$. Thus, $(u^{*\alpha})^* = -u^\alpha$, where we have used the fact that $C = \gamma^0$ is imaginary. Similarly $(u^\alpha)^* = -u^{*\alpha}$. Likewise for v. Care should be taken while complex conjugating dot products of spinors, for instance $(v^*(\mathbf{p}_i)v^*(\mathbf{p}_j))^* = -(v(\mathbf{p}_i)v(\mathbf{p}_j))$, $(u(\mathbf{p}_i)u(\mathbf{p}_j))^* = -(u^*(\mathbf{p}_i)u^*(\mathbf{p}_j))$, and so on.

Let us define

$$\sqrt{ae^{i\alpha}} = |\sqrt{a}|e^{i\frac{\alpha}{2}}.$$

Clearly our function is single valued only on a double cover of the complex plane. In other words our square root function is well defined if α is specified modulo 4π, but is not well defined if α is specified modulo 2π. We define

$$\begin{aligned}
u(\mathbf{p},\alpha) = u(e^{i\alpha}p^\mu) &= \begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}} + ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix}, \\
v(\mathbf{p},\alpha) = v(e^{i\alpha}p^\mu) &= -\begin{pmatrix} -e^{i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{i\frac{\alpha}{2}} - ime^{-i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix}, \\
u^*(\mathbf{p},\alpha) &= \begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}} - ime^{i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix}, \\
v^*(\mathbf{p},\alpha) &= -\begin{pmatrix} -e^{-i\frac{\alpha}{2}}\sqrt{p^0-p^1} \\ \frac{p^3 e^{-i\frac{\alpha}{2}} + ime^{+i\frac{\alpha}{2}}}{\sqrt{p^0-p^1}} \end{pmatrix},
\end{aligned} \tag{2.24}$$

with $\alpha \in [0, 4\pi)$. It follows immediately from these definitions that

$$\begin{aligned}
u(\mathbf{p},\alpha+\pi) &= -iv(\mathbf{p},\alpha), & v(\mathbf{p},\alpha+\pi) &= -iu(\mathbf{p},\alpha), \\
u(\mathbf{p},\alpha-\pi) &= iv(\mathbf{p},\alpha), & v(\mathbf{p},\alpha-\pi) &= iu(\mathbf{p},\alpha), \\
u^*(\mathbf{p},\alpha) &= -v(\mathbf{p},-\alpha), & v^*(\mathbf{p},\alpha) &= -u(\mathbf{p},-\alpha).
\end{aligned} \tag{2.25}$$

Notice, in particular, that the choice $\alpha = \pi$ and $\alpha = -\pi$ both amount to the replacement of p^μ with $-p^\mu$. Note also that the complex conjugation of $u(p,\alpha)$ is equal to the function $u^*(p)$ with p rotated by $-\alpha$.

2.4 Constraints of supersymmetry on scattering

In this paper we will study 2×2 scattering of particles in an $\mathcal{N} = 1$ supersymmetric field theory. In this subsection we set up our conventions and notations and explore the constraints of supersymmetry on scattering amplitudes.

Let us consider the scattering process

$$1 + 2 \to 3 + 4 \tag{2.26}$$

where $1, 2$ represent initial state particles and $3, 4$ are final state particles. Let the i^{th} particle be associated with the superfield Φ_i. As a scattering amplitude represents the transition between free incoming and free outgoing onshell particles, the initial and final states of Φ_i are effectively subject to the free equation of motion

$$\left(D^2 + m_i\right)\Phi_i = 0 \tag{2.27}$$

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where $D^2 = \frac{1}{2}D^\alpha D_\alpha$. The general solution to this free equation of motion is

$$\Phi(x,\theta) = \int \frac{d^2p}{\sqrt{2p^0}(2\pi)^2}\Bigg[\Big(a(\mathbf{p})(1+m\theta^2) + \theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big)e^{ip.x} \\ + \Big(a^{c\dagger}(\mathbf{p})(1+m\theta^2) + \theta^\alpha v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})\Big)e^{-ip.x}\Bigg] \tag{2.28}$$

where $a/a^\dagger$ are annihilation/creation operator for the bosonic particles and $\alpha/\alpha^\dagger$ are annihilation/creation operators for the fermionic particles respectively.[12] The bosonic and fermionic oscillators obey the commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2\delta^2(\mathbf{p}-\mathbf{p}'), \quad [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = (2\pi)^2\delta^2\left(\mathbf{p}-\mathbf{p}'\right). \tag{2.29}$$

(a^c and α^c obey analogous commutation relations).

The action of the supersymmetry operator on a free onshell superfield is simple

$$[Q_\alpha, \Phi_i] = \\ Q_\alpha\Phi_i = i\int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}}\Bigg[\Big(u_\alpha(\mathbf{p})(1+m\theta^2)\alpha(\mathbf{p}) + \theta^\beta(-u_\beta(\mathbf{p})u^*_\alpha(\mathbf{p}))a(\mathbf{p})\Big)e^{ip.x} \\ + \Big(v_\alpha(\mathbf{p})(1+m\theta^2)\alpha^{c\dagger}(\mathbf{p}) + \theta^\beta(v_\beta(\mathbf{p})v^*_\alpha(\mathbf{p}))a^{c\dagger}(\mathbf{p})\Big)e^{-ip.x}\Bigg]. \tag{2.30}$$

In other words, the action of the supersymmetry generator on onshell superfields is given by

$$-iQ_\alpha = u_\alpha(\mathbf{p}_i)\left(a\partial_\alpha + a^c\partial_{\alpha^c}\right) + u^*_\alpha(\mathbf{p}_i)\left(-\alpha\partial_a + \alpha^c\partial_{a^c}\right) \\ + v_\alpha(\mathbf{p}_i)\left(a^\dagger\partial^\dagger_\alpha + (a^c)^\dagger\partial_{(\alpha^c)^\dagger}\right) + v^*_\alpha(\mathbf{p}_i)\left(\alpha^\dagger\partial^\dagger_a + (\alpha^c)^\dagger\partial_{(a^c)^\dagger}\right). \tag{2.31}$$

The explicit action of Q_α on onshell superfields may be repackaged as follows. Let us define a superfield of annihilation operators, and another superfield for creation operators:

$$A_i(\mathbf{p}) = a_i(\mathbf{p}) + \alpha_i(\mathbf{p})\theta_i\,, \\ A^\dagger_i(\mathbf{p}) = a^\dagger_i(\mathbf{p}) + \theta_i\alpha^\dagger_i(\mathbf{p})\,. \tag{2.32}$$

Here θ_i is a new formal superspace parameter (θ_i has nothing to do with the θ_α that appear in the superfield action (2.1)). It follows from (2.31) and (2.32) that

$$[Q_\alpha, A_i(\mathbf{p}_i,\theta_i)] = Q^1_\alpha A_i(\mathbf{p}_i,\theta_i) \\ [Q_\alpha, A^\dagger_i(\mathbf{p}_i,\theta_i)] = Q^2_\alpha A^\dagger_i(\mathbf{p}_i,\theta_i) \tag{2.33}$$

where

$$Q^1_\beta = i\left(-u_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\beta(\mathbf{p})\theta\right) \\ Q^2_\beta = i\left(v_\beta(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + v^*_\beta(\mathbf{p})\theta\right). \tag{2.34}$$

[12] Similarly $a^c/a^{c\dagger}$ and $\alpha^c/\alpha^{c\dagger}$ are the annihilation/creation operators for the bosonic and fermionic anti-particles respectively.

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We are interested in the S matrix

$$\begin{aligned}S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)\sqrt{(2p_1^0)(2p_2^0)(2p_3^0)(2p_4^0)} =&\\ \langle 0|A_4(\mathbf{p}_4,\theta_4)A_3(\mathbf{p}_3,\theta_3)UA_2^\dagger(\mathbf{p}_2,\theta_2)A_1^\dagger(\mathbf{p}_1,\theta_1)|0\rangle&\end{aligned} \tag{2.35}$$

where U is an evolution operator (the r.h.s. denotes the transition amplitude from the in state with particles 1 and 2 to the out state with particles 3 and 4).

The condition that the S matrix defined in (2.35) is invariant under supersymmetry follows from the action of supersymmetries on oscillators given in (2.30). The resultant equation for the S matrix may be written in terms of the operators defined in (2.34) as

$$\begin{aligned}\Big(&\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,\theta_1)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_2,\theta_2)\\ &+\overrightarrow{Q}^2_\alpha(\mathbf{p}_3,\theta_3)+\overrightarrow{Q}^2_\alpha(\mathbf{p}_4,\theta_4)\Big)S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=0\,.\end{aligned} \tag{2.36}$$

We have explicitly solved (2.36); the solution[13] is given by

$$\begin{aligned}S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) &= \mathcal{S}_B+\mathcal{S}_F\;\theta_1\theta_2\theta_3\theta_4+\left(\frac{1}{2}C_{12}\mathcal{S}_B-\frac{1}{2}C^*_{34}\mathcal{S}_F\right)\theta_1\theta_2\\ &+\left(\frac{1}{2}C_{13}\mathcal{S}_B-\frac{1}{2}C^*_{24}\mathcal{S}_F\right)\theta_1\theta_3+\left(\frac{1}{2}C_{14}\mathcal{S}_B+\frac{1}{2}C^*_{23}\mathcal{S}_F\right)\theta_1\theta_4+\left(\frac{1}{2}C_{23}\mathcal{S}_B+\frac{1}{2}C^*_{14}\mathcal{S}_F\right)\theta_2\theta_3\\ &+\left(\frac{1}{2}C_{24}\mathcal{S}_B-\frac{1}{2}C^*_{13}\mathcal{S}_F\right)\theta_2\theta_4+\left(\frac{1}{2}C_{34}\mathcal{S}_B-\frac{1}{2}C^*_{12}\mathcal{S}_F\right)\theta_3\theta_4\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}\frac{1}{2}C_{12}&=-\frac{1}{4m}v^*(\mathbf{p}_1)v^*(\mathbf{p}_2) & \frac{1}{2}C_{23}&=-\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_3)\\ \frac{1}{2}C_{13}&=-\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_3) & \frac{1}{2}C_{24}&=-\frac{1}{4m}v^*(\mathbf{p}_2)u^*(\mathbf{p}_4)\\ \frac{1}{2}C_{14}&=-\frac{1}{4m}v^*(\mathbf{p}_1)u^*(\mathbf{p}_4) & \frac{1}{2}C_{34}&=-\frac{1}{4m}u^*(\mathbf{p}_3)u^*(\mathbf{p}_4)\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}\frac{1}{2}C^*_{12}&=\frac{1}{4m}v(\mathbf{p}_1)v(\mathbf{p}_2) & \frac{1}{2}C^*_{23}&=\frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_3)\\ \frac{1}{2}C^*_{13}&=\frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_3) & \frac{1}{2}C^*_{24}&=\frac{1}{4m}v(\mathbf{p}_2)u(\mathbf{p}_4)\\ \frac{1}{2}C^*_{14}&=\frac{1}{4m}v(\mathbf{p}_1)u(\mathbf{p}_4) & \frac{1}{2}C^*_{34}&=\frac{1}{4m}u(\mathbf{p}_3)u(\mathbf{p}_4)\end{aligned} \tag{2.39}$$

[13]The superspace S matrix (2.37) encodes different processes allowed by supersymmetry in the theory. In particular, the presence of grassmann parameters indicates fermionic in (θ_1,θ_2) and fermionic out (θ_3,θ_4) states. The absence of grassmann parameter indicates a bosonic in/out state. Thus, the no θ term $\mathcal{S}_B$ encodes the $2\to 2$ S matrix for a purely bosonic process, while the four θ term $\mathcal{S}_F$ encodes the $2\to 2$ S matrix of a purely fermionic process. Note in particular that S matrices corresponding to all other $2\to 2$ processes that involve both bosons and fermions are completely determined in terms of the S matrices $\mathcal{S}_B$ and $\mathcal{S}_F$ together with (2.38) and (2.39).

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Note that the general solution to (2.36) is given in terms of two arbitrary functions $\mathcal{S}_B$ and $\mathcal{S}_F$ of the four momenta; (2.36) determines the remaining six functions in the general expansion of the S matrix in terms of these two functions. See appendix B for a check of these relations from another viewpoint (involving offshell supersymmetry of the effective action, see section 3.4).

Although we are principally interested in $\mathcal{N}=1$ supersymmetric theories in this paper, we will sometimes study the special limit $w=1$ in which (2.1) enjoys an enhanced $\mathcal{N}=2$ supersymmetry. In this case the additional supersymmetry further constrains the S matrix. In appendix C we demonstrate that the additional supersymmetry determines $\mathcal{S}_B$ in terms of $\mathcal{S}_F$. In the $\mathcal{N}=2$ case, in other words, all components of the S matrix are determined by supersymmetry in terms of the four boson scattering matrix.

2.5 Supersymmetry and dual supersymmetry

The strong weak coupling duality we study in this paper is conjectured to be a Bose-Fermi duality. In other words

$$a^D = \alpha, \quad \alpha^D = a \tag{2.40}$$

together with a similar exchange of bosons and fermions for creation operators (the superscript D stands for 'dual'). Suppose we define

$$\begin{aligned} A_i^D(\mathbf{p}) &= a_i^D(\mathbf{p}) + \alpha_i^D(\mathbf{p})\theta_i\,, \\ (A^D)_i^\dagger(\mathbf{p}) &= (a^D)_i^\dagger(\mathbf{p}) + \theta_i(\alpha_i^D)^\dagger(\mathbf{p})\,. \end{aligned} \tag{2.41}$$

The dual supersymmetries must act in the same way on A^D and $(A^D)^\dagger$ as ordinary supersymmetries act on A and A^D. In other words the action of dual supersymmetries on A^D and $(A^D)^\dagger$ is given by

$$\begin{aligned} [Q_\alpha^D, A_i^D(\mathbf{p}_i,\theta_i)] &= (Q^D)^1_\alpha A_i^D(\mathbf{p}_i,\theta_i)\,, \\ [Q_\alpha^D, (A^D)_i^\dagger(\mathbf{p}_i,\theta_i)] &= (Q^D)^2_\alpha (A^D)_i^\dagger(\mathbf{p}_i,\theta_i)\,, \end{aligned} \tag{2.42}$$

where

$$\begin{aligned} (Q^D)^1_\beta &= i\left(-u_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - v_\beta(\mathbf{p},-m)\theta\right)\,, \\ (Q^D)^2_\beta &= i\left(v_\beta(\mathbf{p},-m)\frac{\overrightarrow{\partial}}{\partial\theta} - u_\beta(\mathbf{p},-m)\theta\right)\,. \end{aligned} \tag{2.43}$$

The spinors in (2.43) are all evaluated at $-m$ as duality flips the sign of the pole mass.

The action of the dual supersymmetries on A and $A^\dagger$ is obtained from (2.43) upon performing the interchange $\theta \leftrightarrow \partial_\theta$ (this accounts for the interchange of bosons and fermions). Using also (2.22) we find that

$$\begin{aligned} [Q_\alpha^D, A_i(\mathbf{p}_i,\theta_i)] &= -Q^1_\alpha A_i^D(\mathbf{p}_i,\theta_i)\,, \\ [Q_\alpha^D, A_i^\dagger(\mathbf{p}_i,\theta_i)] &= Q^2_\alpha (A^D)_i^\dagger(\mathbf{p}_i,\theta_i)\,. \end{aligned} \tag{2.44}$$

It follows, in particular, that an S matrix invariant under the usual supersymmetries is automatically invariant under dual supersymmetries. In other words onshell supersymmetry 'commutes' with duality.

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2.6 Naive crossing symmetry and supersymmetry

Let us define the analytically rotated supersymmetry operators[14]

$$
\begin{aligned}
Q^1_\beta(\mathbf{p},\alpha,\theta) &= i\left(-u_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta}+u^*_\beta(\mathbf{p},-\alpha)\theta\right),\\
Q^2_\beta(\mathbf{p},\alpha,\theta) &= i\left(v_\beta(\mathbf{p},\alpha)\frac{\overrightarrow{\partial}}{\partial\theta}+v^*_\beta(\mathbf{p},-\alpha)\theta\right).
\end{aligned}
\tag{2.45}
$$

It is easily verified from these definitions that

$$
Q^2_\alpha(\mathbf{p},0,-i\theta) = Q^1_\alpha(\mathbf{p},\pi,\theta)\,. \tag{2.46}
$$

Using (2.46) the equation (2.36) may equivalently be written as

$$
\begin{aligned}
&\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,0,\theta_1)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_2,0,\theta_2)\\
&\quad+\overrightarrow{Q}^1_\alpha(\mathbf{p}_3,\pi,\theta_3)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_4,\pi,\theta_4)\Big)S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,-i\theta_3,\mathbf{p}_4,-i\theta_4)=0
\end{aligned}
\tag{2.47}
$$

with $p_1+p_2=p_3+p_4$.

The constraints of supersymmetry on the S matrix are consistent with (naive) crossing symmetry. In order to make this manifest, we define a 'master' function S_M

$$
S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4)\,.
$$

The master function S_M is defined so that

$$
S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,-i\theta_3,\mathbf{p}_4,-i\theta_4) = S_M(\mathbf{p}_1,0,\theta_1,\mathbf{p}_2,0,\theta_2,\mathbf{p}_3,\pi,\theta_3,\mathbf{p}_4,\pi,\theta_4)\,. \tag{2.48}
$$

In other words S_M is S with the replacement $-i\theta_3\to\theta_3$, $-i\theta_4\to\theta_4$, analytically rotated to general values of the phase ϕ_1, ϕ_2, ϕ_3 and ϕ_4. It follows from (2.47) that the master equation S_M obeys the completely symmetrical supersymmetry equation

$$
\begin{aligned}
&\Big(\overrightarrow{Q}^1_\alpha(\mathbf{p}_1,\phi_1,\theta_1)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_2,\phi_2,\theta_2)+\overrightarrow{Q}^1_\alpha(\mathbf{p}_3,\phi_3,\theta_3)\\
&\quad+\overrightarrow{Q}^1_\alpha(\mathbf{p}_4,\phi_4,\theta_4)\Big)S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4)=0\,.
\end{aligned}
\tag{2.49}
$$

The function S_M encodes the scattering matrices in all channels. In order to extract the S matrix for $p_i+p_j\to p_k+p_m$ with $p_i+p_j=p_k+p_m$ (with (i, j, k, m) being any permutation of (1, 2, 3, 4)) we simply evaluate the function S_M with ϕ_i and ϕ_j set to zero, ϕ_k and ϕ_m set to π, θ_i and θ_j left unchanged and θ_k and θ_m replaced by $i\theta_k$ and $i\theta_m$. The fact that the master equation obeys an equation that is symmetrical in the labels $1,2,3,4$ is the statement of (naive) crossing symmetry.

[14]Note that the notation $u^*_\beta(\mathbf{p},-\alpha)$ means that the analytically rotated function of u^* in (2.24) is evaluated at the phase $-\alpha$.

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The solution to the differential equation (2.49) is

$$
\begin{aligned}
S_M(\mathbf{p}_1,\phi_1,\theta_1,\mathbf{p}_2,\phi_2,\theta_2,\mathbf{p}_3,\phi_3,\theta_3,\mathbf{p}_4,\phi_4,\theta_4) &= \tilde{\mathcal{S}_B} + \tilde{\mathcal{S}_F}\theta_1\theta_2\theta_3\theta_4 \\
&+ \frac{\tilde{\mathcal{S}_B}}{4}\sum_{i,j=1}^{4} D_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)\theta_i\theta_j - \frac{\tilde{\mathcal{S}_F}}{8}\sum_{i,j,k,l=1}^{4}\epsilon^{ijkl}\tilde{D}_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j)\theta_k\theta_l
\end{aligned}
\tag{2.50}
$$

where

$$
\begin{aligned}
\frac{1}{2}D_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j) &= -\frac{1}{4m}u^*(\mathbf{p}_i,-\phi_i)u^*(\mathbf{p}_j,-\phi_j)\,, \\
\frac{1}{2}\tilde{D}_{ij}(\mathbf{p}_i,\phi_i,\mathbf{p}_j,\phi_j) &= \frac{1}{4m}u(\mathbf{p}_i,\phi_i)u(\mathbf{p}_j,\phi_j)\,.
\end{aligned}
\tag{2.51}
$$

In the above equations '$*$' means complex conjugation and the spinor indices are contracted from NW-SE as usual. To summarize, S_M obeys the supersymmetric ward identity and is completely solved in terms of two analytic functions $\tilde{\mathcal{S}_B}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$ and $\tilde{\mathcal{S}_F}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)$ of the momenta.

As we have explained under (2.49), the S matrix corresponding to scattering processes in any given channel can be simply extracted out of S_M. For example, let S denote the the S matrix in the channel with p_1, p_2 as in-states and p_3, p_4 as out-states. Then

$$
S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4) = S_M(\mathbf{p}_1,\pi,i\theta_1,\mathbf{p}_2,\pi,i\theta_2,\mathbf{p}_3,0,\theta_3,\mathbf{p}_4,0,\theta_4)\,. \tag{2.52}
$$

It is easily verified that (2.51) together with (2.25) imply (2.38).

Notice that (2.52) maps $\tilde{\mathcal{S}_B}$ to $\mathcal{S}_B$ while $\tilde{\mathcal{S}_F}$ is mapped to $-\mathcal{S}_F$.[15] The minus sign in the continuation of $\mathcal{S}_F$ has an interesting explanation. The four fermion amplitude $\mathcal{S}_F$ has a phase ambiguity. This ambiguity follows from the fact that $\mathcal{S}_F$ is the overlap of initial and final fermions states. These initial and final states are written in terms of the spinors u_α and v_α, which are defined as appropriately normalized solutions of the Dirac equation are inherently ambiguous upto a phase. It is easily verified that the quantity

$$
\left(u^*(\mathbf{p}_1,-\phi_1)u(\mathbf{p}_3,\phi_3)\right)\left(u^*(\mathbf{p}_2,-\phi_2)u(\mathbf{p}_4,\phi_4)\right)
$$

has the same phase ambiguity as $\mathcal{S}_F$. If we define an auxiliary quantity $\tilde{\mathcal{S}_f}$ by the equation

$$
\tilde{\mathcal{S}_F} = -\frac{1}{4m^2}\left(u^*(\mathbf{p}_1,-\phi_1)u(\mathbf{p}_3,\phi_3)\right)\left(u^*(\mathbf{p}_2,-\phi_2)u(\mathbf{p}_4,\phi_4)\right)\tilde{\mathcal{S}_f} \tag{2.53}
$$

and $\mathcal{S}_f$ by

$$
\mathcal{S}_F = -\frac{1}{4m^2}\left(u^*(\mathbf{p}_1)u(\mathbf{p}_3)\right)\left(u^*(\mathbf{p}_2)u(\mathbf{p}_4)\right)\mathcal{S}_f\,, \tag{2.54}
$$

then the phases of $\mathcal{S}_f$ and $\tilde{\mathcal{S}_f}$ are unambiguous and so potentially physical. As the quantity

$$
\left(u^*(\mathbf{p}_1,-\phi_1)u(\mathbf{p}_3,\phi_3)\right)\left(u^*(\mathbf{p}_2,-\phi_2)u(\mathbf{p}_4,\phi_4)\right)
$$

picks up a minus sign under the phase rotation that takes us from S_M to S. It follows that $\tilde{\mathcal{S}_f}$ rotates to $\mathcal{S}_f$ with no minus sign.

[15] Of course $\tilde{\mathcal{S}_B}$ and $\tilde{\mathcal{S}_F}$ are evaluated at $\phi_1 = \phi_2 = \pi$ while $\mathcal{S}_B$ and $\mathcal{S}_F$ are evaluated at $\phi_1 = \phi_2 = 0$; roughly speaking this amounts to the replacement $p_1^\mu \to -p_1^\mu$, $p_2^\mu \to -p_2^\mu$.

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2.7 Properties of the convolution operator

Like any matrices, S matrices can be multiplied. The multiplication rule for two S matrices, S_1 and S_2, expressed as functions in onshell superspace is given by

$$S_1 \star S_2 \equiv \int d\Gamma S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1)$$
$$2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \tag{2.55}$$

where the measure $d\Gamma$ is

$$d\Gamma = \frac{d^2k_3}{2k_3^0(2\pi)^2} \frac{d^2k_4}{2k_4^0(2\pi)^2} \frac{d^2k_1}{2k_1^0(2\pi)^2} \frac{d^2k_2}{2k_2^0(2\pi)^2} d\phi_1 d\phi_3 d\phi_2 d\phi_4 \, . \tag{2.56}$$

It is easily verified that the onshell superfield I

$$I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \exp(\theta_1\theta_3 + \theta_2\theta_4) I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$$
$$I(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 2p_3^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3) 2p_4^0 (2\pi)^2 \delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4) \tag{2.57}$$

is the identity operator under this multiplication rule, i.e.

$$S \star I = I \star S = S \tag{2.58}$$

for any S. It may be verified that I defined in (2.57) obeys (2.36) and so is supersymmetric.

In appendix D we demonstrate that if S_1 and S_2 are onshell superfields that obey (2.36), then $S_1 \star S_2$ also obeys (2.36). In other words the product of two supersymmetric S matrices is also supersymmetric.

The onshell superfield corresponding to $S^\dagger$ is given in terms of the onshell superfield corresponding to S by the equation

$$S^\dagger(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2) \, . \tag{2.59}$$

The equation satisfied by $S^\dagger$ can be obtained by complex conjugating and interchanging the momenta in the supersymmetry invariance condition for S (see (D.6)). It follows from the anti-hermiticity of Q that

$$\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)} \right) S^*(\mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4, \mathbf{p}_1, \theta_3, \mathbf{p}_2, \theta_4) = 0 \tag{2.60}$$

which implies $[Q, S^\dagger] = 0$. Thus $S^\dagger$ is supersymmetric if and only if S is supersymmetric.

2.8 Unitarity of scattering

The unitarity condition

$$S S^\dagger = \mathbb{I} \tag{2.61}$$

may be rewritten in the language of onshell superfields as[16]

$$(S \star S^\dagger - I) = 0 \, . \tag{2.62}$$

[16] As explained in [36], the unitarity equation for 2×2 does not receive contributions from $2 \times n$ scattering in the large N limits studied in the current paper as well.

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It follows from the general results of the previous subsection that the l.h.s. of (2.62) is supersymmetric, i.e it obeys (2.36). Recall that any onshell superfield that obeys (2.36) must take the form (2.37) where $\mathcal{S}_B$ and $\mathcal{S}_F$ are the zero theta and 4 theta terms in the expansion of the corresponding object. In particular, in order to verify that the l.h.s. of (2.62) vanishes, it is sufficient to verify that its zero and 4 theta components vanish.

Inserting the explicit solutions for S and $S^\dagger$, one finds that the no-theta term of (2.62) is proportional to (we have used that $k_3 \cdot k_4 = p_3 \cdot p_4$ onshell)

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\Big[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&-\frac{1}{16m^2}\left(2(p_3\cdot p_4+m^2)w\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right.\\
&+u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+v(\mathbf{p}_1)v(\mathbf{p}_2)\ u(\mathbf{k}_3)u(\mathbf{k}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&\left.+v(\mathbf{p}_1)v(\mathbf{p}_2)\ v^*(\mathbf{p}_3)v^*(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\Big]\\
&\quad= 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\,.
\end{aligned} \tag{2.63}$$

The four theta term in (2.62) is proportional to

$$\begin{aligned}
&\int \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\Big[-\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+\frac{1}{16m^2}\left(2(p_3\cdot p_4+m^2)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right.\\
&+u(\mathbf{k}_3)u(\mathbf{k}_4)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ u^*(\mathbf{k}_3)u^*(\mathbf{k}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\
&\left.+v^*(\mathbf{p}_1)v^*(\mathbf{p}_2)\ v(\mathbf{p}_3)v(\mathbf{p}_4)\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\Big]\\
&\quad= -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)\,.
\end{aligned} \tag{2.64}$$

The equations (2.63) and (2.64) are necessary and sufficient to ensure unitarity.

(2.63) and (2.64) may be thought of as constraints imposed by unitarity on the four boson scattering matrix $\mathcal{S}_B$ and the four fermion scattering matrix $\mathcal{S}_F$. These conditions are written in terms of the onshell spinors u and v (rather than the momenta of the scattering particles for a reason we now pause to review. Recall that the Dirac equation and normalization conditions define u_α and v_α only upto an undetermined phase (which could be a function of momentum). An expression built out of u's and v's can be written unambiguously in terms of onshell momenta if and only if all undetermined phases cancel out. The phases of terms involving $\mathcal{S}_F$ in (2.63) and (2.64) do not cancel. This might at first appear to be a paradox; surely the unitarity (or lack) of an S matrix cannot depend on the unphysical choice of an arbitrary phase. The resolution to this 'paradox' is simple; the function $\mathcal{S}_F$ is itself not phase invariant, but transforms under phase transformations like $(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))$. It is thus useful to define

$$\mathcal{S}_F = \frac{1}{4m^2}\,(u(\mathbf{p}_1)u(\mathbf{p}_2))\,(v(\mathbf{p}_3)v(\mathbf{p}_4))\,\mathcal{S}_f\,. \tag{2.65}$$

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The utility of this definition is that $\mathcal{S}_f$ does not suffer from a phase ambiguity. Rewritten in terms of $\mathcal{S}_B$ and $\mathcal{S}_f$, the unitarity equations may be written entirely in terms of participating momenta (with no spinors).[17] In terms of the quantity

$$Y(\mathbf{p}_3, \mathbf{p}_4) = \frac{2(p_3 \cdot p_4 + m^2)}{16m^2} \tag{2.66}$$

and

$$d\Gamma' = \frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}$$

$$\begin{aligned}
\int d\Gamma' \Big[& \mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
& - Y(\mathbf{p}_3, \mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_1, \mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\Big) \\
& \Big(\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_3, \mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\Big)\Big] \\
& = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4)
\end{aligned} \tag{2.67}$$

and

$$\begin{aligned}
\int d\Gamma' \Big[& - 16Y^2(\mathbf{p}_3, \mathbf{p}_4)\mathcal{S}_f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\mathcal{S}_f^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) \\
& + Y(\mathbf{p}_3, \mathbf{p}_4)\Big(\mathcal{S}_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_1, \mathbf{p}_2)\mathcal{S}_f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_3, \mathbf{k}_4)\Big) \\
& \Big(\mathcal{S}_B^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4) + 4Y(\mathbf{p}_3, \mathbf{p}_4)\mathcal{S}_f^*(\mathbf{p}_3, \mathbf{p}_4, \mathbf{k}_3, \mathbf{k}_4)\Big)\Big] \\
& = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1 - \mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2 - \mathbf{p}_4).
\end{aligned} \tag{2.68}$$

The equations (2.67) and (2.68) followed from (2.61). It is useful to rephrase the above equations in terms of the "T matrix" that represents the actual interacting part of the "S matrix". Using the definition of the Identity operator (2.57) we can write a superfield expansion to define the "T matrix" as

$$\begin{aligned}
S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \theta_3, \mathbf{k}_4, \theta_4) = & \, I(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \theta_3, \mathbf{k}_4, \theta_4) \\
& + i(2\pi)^3\delta^3(p_1 + p_2 - p_3 - p_4)T(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \theta_3, \mathbf{k}_4, \theta_4)\,.
\end{aligned} \tag{2.69}$$

The identity operator is defined in (2.57) is a supersymmetry invariant. It follows that the " T matrix" is also invariant under supersymmetry. In other words the "T matrix" obeys (2.36) and has a superfield expansion[18]

$$\begin{aligned}
T(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = & \, \mathcal{T}_B + \mathcal{T}_F \; \theta_1\theta_2\theta_3\theta_4 + \left(\frac{1}{2}C_{12}\mathcal{T}_B - \frac{1}{2}C_{34}^*\mathcal{T}_F\right)\theta_1\theta_2 \\
& + \left(\frac{1}{2}C_{13}\mathcal{T}_B - \frac{1}{2}C_{24}^*\mathcal{T}_F\right)\theta_1\theta_3 + \left(\frac{1}{2}C_{14}\mathcal{T}_B + \frac{1}{2}C_{23}^*\mathcal{T}_F\right)\theta_1\theta_4 + \left(\frac{1}{2}C_{23}\mathcal{T}_B + \frac{1}{2}C_{14}^*\mathcal{T}_F\right)\theta_2\theta_3 \\
& + \left(\frac{1}{2}C_{24}\mathcal{T}_B - \frac{1}{2}C_{13}^*\mathcal{T}_F\right)\theta_2\theta_4 + \left(\frac{1}{2}C_{34}\mathcal{T}_B - \frac{1}{2}C_{12}^*\mathcal{T}_F\right)\theta_3\theta_4
\end{aligned} \tag{2.70}$$

[17] See section E for a derivation of this result.

[18] The matrices $\mathcal{T}_B$ and $\mathcal{T}_F$ correspond to the T matrices of the four boson and four fermion scattering respectively.

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where

$$\mathcal{T}_F = \frac{1}{4m^2}\left(u(\mathbf{p}_1)u(\mathbf{p}_2)\right)\left(v(\mathbf{p}_3)v(\mathbf{p}_4)\right)\mathcal{T}_f \tag{2.71}$$

and the coefficients C_{ij} are given as before in (2.38) and (2.39).

It follows from (2.69) that

$$\begin{aligned}\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,,\\ \mathcal{S}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,.\end{aligned} \tag{2.72}$$

Substituting the definitions (2.72) into (2.67) and (2.68) the unitarity conditions can be rewritten as

$$\begin{aligned}\int d\tilde{\Gamma}\Big[&\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\ &- Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4) + 4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\ &\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) + 4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]\\ &\quad = i(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) - \mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2))\end{aligned} \tag{2.73}$$

and

$$\begin{aligned}\int d\tilde{\Gamma}\Big[&- 16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\\ &+ Y(\mathbf{p}_3,\mathbf{p}_4)\Big(\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4) + 4Y(\mathbf{p}_1,\mathbf{p}_2)\mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\Big)\\ &\Big(\mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) + 4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Big)\Big]\\ &\quad = 4iY(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{T}_f^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2) - \mathcal{T}_f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\right)\end{aligned} \tag{2.74}$$

where

$$d\tilde{\Gamma} = (2\pi)^3\delta^3(p_1+p_2-k_3-k_4)\frac{d^2k_3}{2k_3^0(2\pi)^2}\frac{d^2k_4}{2k_4^0(2\pi)^2}\,.$$

The equations (2.73) and (2.74) can be put in a more user friendly form by going to the center of mass frame with the definition

$$\begin{aligned}p_1 &= \left(\sqrt{p^2+m^2}, p, 0\right), & p_2 &= \left(\sqrt{p^2+m^2}, -p, 0\right)\\ p_3 &= \left(\sqrt{p^2+m^2}, p\cos(\theta), p\sin(\theta)\right), & p_4 &= \left(\sqrt{p^2+m^2}, -p\cos(\theta), -p\sin(\theta)\right)\end{aligned} \tag{2.75}$$

where θ is the scattering angle between p_1 and p_3. In terms of the Mandelstam variables

$$\begin{aligned}s &= -(p_1+p_2)^2\,, & t &= -(p_1-p_3)^2, & u &= (p_1-p_4)^2, \quad s+t+u = 4m^2\,,\\ s &= 4(p^2+m^2)\,, & t &= -2p^2(1-\cos(\theta))\,, & u &= -2p^2(1+\cos(\theta))\,.\end{aligned} \tag{2.76}$$

Using the definitions we see that (2.66) becomes

$$Y = \frac{2(p_3\cdot p_4 + m^2)}{16m^2} = \frac{-s+4m^2}{16m^2} = Y(s)\,. \tag{2.77}$$

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Then (2.73) and (2.74) can be put in the form (See for instance eq. 2.58–eq. 2.59 of [36])

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B(s,\theta)+4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) \\ +\mathcal{T}_B(s,\theta)\mathcal{T}_B^*(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^*(s,-\alpha)-\mathcal{T}_B(s,\alpha)) \tag{2.78}$$

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B(s,\theta)+4Y(s)\mathcal{T}_f(s,\theta))(\mathcal{T}_B^*(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^*(s,-(\alpha-\theta))) \\ -16Y(s)^2\mathcal{T}_f(s,\theta)\mathcal{T}_f^*(s,-(\alpha-\theta))\Big)=i4Y(s)(-\mathcal{T}_f(s,\alpha)+\mathcal{T}_f^*(s,-\alpha))\,. \tag{2.79}$$

In a later section 4 we will use the simplified equations (2.78) and (2.79) for the unitarity analysis.

3 Exact computation of the all orders S matrix

In this section we will present results and conjectures for the the 2×2 S matrix of the general $\mathcal{N}=1$ theory (3.2) at all orders in the t'Hooft coupling. In section 3.2 we recall the action for our theory and determine the bare propagators for the scalar and vector superfields. At leading order in the $\frac{1}{N}$ the vector superfield propagator is exact (it is not renormalized). However the propagator of the scalar superfield does receive corrections. In section 3.3, we determine the all orders propagator for the superfield Φ by solving the relevant Schwinger-Dyson equation. We will then turn to the determination of the exact offshell four point function of the superfield Φ. As in [36], we demonstrate that this four point function is the solution to a linear integral equation which we explicitly write down in section 3.5. In a particular kinematic regime we present an exact solution to this integral equation in section 3.6. In order to obtain the S matrix, in section 3.7 we take the onshell limit of this answer. The kinematic restriction on our offshell result turns out to be inconsistent with the onshell limit in one of the four channels of scattering (particle-antiparticle scattering in the singlet channel) and so we do not have an explicit computation of the S matrix in this channel. In the other three channels, however, we are able to extract the full S matrix (with no kinematic restriction) albeit in a particular Lorentz frame. In section 3.7 we present the unique covariant expressions for the S matrix consistent with our results. In section 3.8 we report our result that the covariant S matrix reported in section 3.7 is duality invariant. We present explicit exact results for the S matrices in the T and U channels of scattering in section 3.9. In section 3.10 we present the explicit conjecture for the S matrix in the singlet (S) channel. In section 3.11 we report the explicit S matrices for the $\mathcal{N}=2$ theory.

3.1 Supersymmetric light cone gauge

We study the general $\mathcal{N}=1$ theory (2.1). Wess-Zumino gauge, employed in subsection 2.1 to display the physical content of our theory, is inconvenient for actual computations as it breaks manifest supersymmetry. In other words if Γ_α is chosen to lie in Wess-Zumino gauge,

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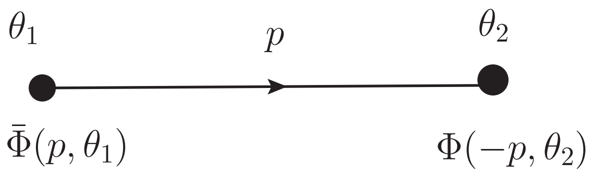

Figure 1. Scalar superfield propagator.

it is in general not the case that $Q_\beta\Gamma_\alpha$ also respects this gauge condition. In all calculations presented in this paper we will work instead in 'supersymmetric light cone gauge'[19]

$$\Gamma_- = 0\,. \tag{3.1}$$

As Γ_- transforms homogeneously under supersymmetry (see (2.6)) it is obvious that this gauge choice is supersymmetric. It is also easily verified that all gauge self interactions in (2.1) vanish in our lightcone gauge and the action (2.1) simplifies to

$$\begin{aligned} S_{\text{tree}} = -\int d^3x d^2\theta \Big[&-\frac{\kappa}{8\pi} Tr(\Gamma^- i\partial_{--}\Gamma^-) - \frac{1}{2}D^\alpha\bar\Phi D_\alpha\Phi - \frac{i}{2}\Gamma^-(\bar\Phi D_-\Phi - D_-\bar\Phi\Phi) \\ &+ m_0\bar\Phi\Phi + \frac{\pi w}{\kappa}(\bar\Phi\Phi)^2\Big]\,. \end{aligned} \tag{3.2}$$

Note in particular that (3.2) is quadratic in Γ_+.

The condition (3.1) implies, in particular, that the component gauge fields in Γ_α obey

$$A_- = A_1 + iA_2 = 0$$

(see appendix F for more details and further discussion about this gauge). In other words the gauge (3.1) is a supersymmetric generalization of ordinary lightcone gauge.

3.2 Action and bare propagators

The bare scalar propagator that follows from (3.2) is

$$\langle\bar\Phi(\theta_1,p)\Phi(\theta_2,-p')\rangle = \frac{D^2_{\theta_1,p} - m_0}{p^2+m_0^2}\delta^2(\theta_1-\theta_2)(2\pi)^3\delta^3(p-p')\,. \tag{3.3}$$

where m_0 is the bare mass. We have chosen the convention for the momentum flow direction to be from $\bar\Phi$ to Φ (see figure 1). Our sign conventions are such that the momenta leaving a vertex have a positive sign. The notation $D^2_{\theta_1,p}$ means that the operator depends on θ_1 and the momentum p, the explicit form for D^2 and some useful formulae are listed in section A.2. The gauge superfield propagator in momentum space is (see figure 2)

$$\langle\Gamma^-(\theta_1,p)\Gamma^-(\theta_2,-p')\rangle = -\frac{8\pi}{\kappa}\frac{\delta^2(\theta_1-\theta_2)}{p_{--}}(2\pi)^3\delta^3(p-p') \tag{3.4}$$

where $p_{--} = -(p_1+ip_2) = -p_-$. Inserting the expansion (2.2) into the l.h.s. of (3.4) and

[19] We would like to thank S. Ananth and W. Siegel for helpful correspondence on this subject.

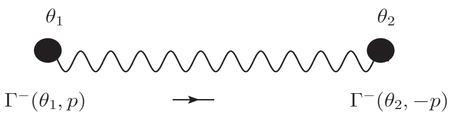

Figure 2. Gauge superfield propagator, the arrow indicates direction of momentum flow.

matching powers of θ, we find in particular that

$$\langle A_+(p)A_3(-p')\rangle = \frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p'),\ \langle A_3(p)A_+(-p')\rangle = -\frac{4\pi i}{\kappa}\frac{1}{p_-}(2\pi)^3\delta^3(p-p'), \quad (3.5)$$

is in perfect agreement with the propagator of the gauge field in regular (non-supersymmetric) lightcone gauge (see appendix A, eq. A.7 of [21])

3.3 The all orders matter propagator

3.3.1 Constraints from supersymmetry

The exact propagator of the matter superfield Φ enjoys invariance under supersymmetry transformations which implies that

$$(Q_{\theta_1,p} + Q_{\theta_2,-p})\langle\bar{\Phi}(\theta_1,p)\Phi(\theta_2,-p)\rangle = 0 \quad (3.6)$$

where the supergenerators $Q_{\theta_1,p}$ were defined in (2.5). This constraint is easily solved. Let the exact scalar propagator take the form

$$\langle\bar{\Phi}(p,\theta_1)\Phi(-p',\theta_2)\rangle = (2\pi)^3\delta^3(p-p')P(\theta_1,\theta_2,p)\,. \quad (3.7)$$

The condition (3.6) implies that the function P obeys the equation

$$\left[\frac{\partial}{\partial\theta_1^\alpha} + \frac{\partial}{\partial\theta_2^\alpha} - p_{\alpha\beta}(\theta_1^\beta - \theta_2^\beta)\right]P(\theta_1,\theta_2,p) = 0\,. \quad (3.8)$$

The most general solution to (3.8) is

$$C_1(p^\mu)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + C_2(p^\mu)\delta^2(\theta_1-\theta_2) \quad (3.9)$$

or equivalently[20]

$$P(\theta_1,\theta_2,p) = \exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)\left(C_1(p^\mu) + C_2(p^\mu)\delta^2(\theta_1-\theta_2)\right) \quad (3.10)$$

where $C_1(p^\mu)$ is an arbitrary function of p^μ of dimension m^{-2}, while $C_2(p^\mu)$ is another function of p^μ of dimension m^{-1}.

It is easily verified using the formulae (A.21) that the bare propagator (3.3) can be recast in the form (3.10) with

$$C_1 = \frac{1}{p^2+m_0^2}\,,\ C_2 = \frac{m_0}{p^2+m_0^2}\,. \quad (3.11)$$

[20]The equivalence of (3.10) and (3.9) follows from the observation that $\theta^a A_{ab}\theta^b$ vanishes if A_{ab} is symmetric in a and b.

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Figure 3. Integral equation for self energy.

In a similar manner supersymmetry constrains the terms quadratic in Φ and $\bar{\Phi}$ in the quantum effective action. In momentum space the most general supersymmetric quadratic effective action takes the form[21]

$$S = -\int \frac{d^3p}{(2\pi)^3} d^2\theta \bar{\Phi}(p,\theta)\left(A(p)D^2 + B(p)\right)\Phi(-p,\theta) \tag{3.12}$$

$$= -\int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \bar{\Phi}(p,\theta_1)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)(A(p)+B(p)\delta^2(\theta_1-\theta_2))\Phi(-p,\theta_2). \tag{3.13}$$

The tree level quadratic action of our theory is clearly of the form (3.12) with $A(p) = 1$ and $B(p) = m_0$.

3.3.2 All orders two point function

Let the exact 1PI quadratic effective action take the form

$$S_2 = \int \frac{d^3p}{(2\pi)^3} d^2\theta_1 d^2\theta_2 \bar{\Phi}(-p,\theta_1)\left(\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta)+m_0\delta^2(\theta_1-\theta_2)+\Sigma(p,\theta_1,\theta_2)\right)\Phi(p,\theta_2)\,. \tag{3.14}$$

It follows from (3.12) that the supersymmetric self energy Σ is of the form

$$\Sigma(p,\theta_1,\theta_2) = C(p)\exp(-\theta_1^\alpha p_{\alpha\beta}\theta_2^\beta) + D(p)\delta^2(\theta_1-\theta_2) \tag{3.15}$$

where $C(p)$ and $D(p)$ are as yet unknown functions of momenta.

Imitating the steps described in section 3 of [21], the self energy Σ defined in (3.14) may be shown to obey the integral equation[22]

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) &= 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \\ &- 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} D_-^{\theta_2,-p} D_-^{\theta_1,p}\left(\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}}P(r,\theta_1,\theta_2)\right) \\ &+ 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} D_-^{\theta_1,r} D_-^{\theta_2,-r} P(r,\theta_1,\theta_2)\end{aligned} \tag{3.16}$$

where $P(p,\theta_1,\theta_2)$ is the exact superfield propagator.[23] Note that the propagator P depends on Σ (in fact P is obtained by inverting quadratic term in effective action (3.14)). In other

[21] In going from the first line to the second line of (3.12) we have integrated by parts and used the identity (A.21). See appendix A.2 for the expressions of superderivatives and operator D^2 in momentum space.

[22] We work at leading order in the large N limit

[23] The first line in the r.h.s. of (3.16) comes from the quartic interaction in figure 3 while the second and third lines in (3.16) comes from the gaugesuperfield interaction in figure 3. Note that each vertex in the diagram corresponding to the gaugesuperfield interaction in figure 3 contains one factor of D, resulting in the two powers of D in the second and third line of (3.16).

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words Σ appears both on the l.h.s. and r.h.s. of (3.16); we need to solve this equation to determine Σ.

Using the equations (A.22), the second and third lines on the r.h.s. if (3.16) may be considerably simplified (see appendix G) and we find

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) &= 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &\quad - 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ &\quad + 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,. \end{aligned} \tag{3.17}$$

Combining the second and third lines on the r.h.s. of (3.17) we see that the factors of p_{--} and r_{--} cancel perfectly between the numerator and denominator, and (3.17) simplifies to

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,. \tag{3.18}$$

Notice that the r.h.s. of (3.18) is independent of p, so it follows that

$$\Sigma(p,\theta_1,\theta_2) = (m-m_0)\delta^2(\theta_1-\theta_2)$$

for some as yet undetermined constant m. It follows that the exact propagator P takes the form of the tree level propagator with m_0 replaced by m i.e.

$$P(p,\theta_1,\theta_2) = \frac{D^2-m}{p^2+m^2}\delta^2(\theta_1-\theta_2)\,. \tag{3.19}$$

Plugging (3.19) into (3.18) and simplifying we find the equation

$$m-m_0 = 2\pi\lambda(w-1)\int \frac{d^3r}{(2\pi)^3}\frac{1}{r^2+m^2}\,. \tag{3.20}$$

The integral on the r.h.s. diverges. Regulating this divergence using dimensional regularization, we find that (3.20) reduces to

$$m-m_0 = \frac{\lambda|m|}{2}(1-w) \tag{3.21}$$

and so

$$m = \frac{2m_0}{2+(-1+w)\lambda\ \mathrm{Sgn(m)}}\,. \tag{3.22}$$

Let us summarize. The *exact* 1PI quadratic effective action for the Φ superfield has the same form as the tree level effective action but with the bare mass m_0 replaced by the exact mass m given in (3.22).[24] As explained in section 2.2 the exact mass (3.22) is duality invariant.

[24] Note that propagator for the fermion in the superfield Φ is the usual propagator for a relativistic fermion of mass m. Recall, of course, that the propagator of Φ is not gauge invariant, and so its form depends on the gauge used in the computation. If we had carried out all computations in Wess-Zumino gauge (which breaks offshell supersymmetry) we would have found the much more complicated expression for the fermion propagator reported in section 2.1 of [33]. Note however that the gauge invariant physical pole mass m of (3.22) agrees perfectly with the pole mass (reported in eq. 1.6 of [33]) of the complicated propagator of [33]. The agreement of gauge invariant quantities in these rather different computations constitutes a nontrivial consistency check of the computations presented in this subsection.

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Note also that the $\mathcal{N} = 2$ point, $w = 1$ there is no renormalization of the mass, and the bare propagator is exact and the bare mass (which equals the pole mass) is itself duality invariant.

3.4 Constraints from supersymmetry on the offshell four point function

Much as with the two point function, the offshell four point function of matter superfields is constrained by the supersymmetric Ward identities. Let us define

$$\begin{aligned}\langle\bar{\Phi}\left(\left(p+q+\frac{l}{4}\right),\theta_1\right)\Phi\left(-p+\frac{l}{4},\theta_2\right)\Phi\left(-(k+q)+\frac{l}{4},\theta_3\right)\bar{\Phi}\left(k+\frac{l}{4},\theta_4\right)\rangle \\ =(2\pi)^3\delta(l)V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k).\end{aligned} \tag{3.23}$$

It follows from the invariance under supersymmetry that

$$(Q_{\theta_1,p+q}+Q_{\theta_2,-p}+Q_{\theta_3,-k-q}+Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)=0\,. \tag{3.24}$$

The general solution to (3.24) is easily obtained (see appendix H.1). Defining

$$\begin{aligned}X &= \sum_{i=1}^{4}\theta_i\,,\\ X_{12} &= \theta_1-\theta_2\,,\\ X_{13} &= \theta_1-\theta_3\,,\\ X_{43} &= \theta_4-\theta_3\,.\end{aligned} \tag{3.25}$$

we find

$$V=\exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right)F(X_{12},X_{13},X_{43},p,q,k)\,. \tag{3.26}$$

where F is an unconstrained function of its arguments. In other words supersymmetry fixes the transformation of V under a uniform shift of all θ parameters $\theta_i \to \theta_i + \gamma$. (for $i = 1 \ldots 4$ where γ is a constant Grassman parameter). The undetermined function F is a function of shift invariant combinations of the four θ_i.

Let us now turn to the structure of the exact 1PI effective action for scalar superfields in our theory. The most general effective action consistent with global $\mathrm{U}(N)$ invariance and supersymmetry takes the form

$$\begin{aligned}S_4=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\theta_4 \\ \left(V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)\Phi_m(-(p+q),\theta_1)\bar{\Phi}^m(p,\theta_2)\bar{\Phi}^n(k+q,\theta_3)\Phi_n(-k,\theta_4)\right)\,.\end{aligned} \tag{3.27}$$

It follows from the definition (3.27) that the function V may be taken to be invariant under the Z_2 symmetry

$$\begin{aligned}&p\to k+q, \quad k\to p+q, \quad q\to -q\,,\\ &\theta_1\to\theta_4, \quad \theta_2\to\theta_3, \quad \theta_3\to\theta_2, \quad \theta_4\to\theta_1\,.\end{aligned} \tag{3.28}$$

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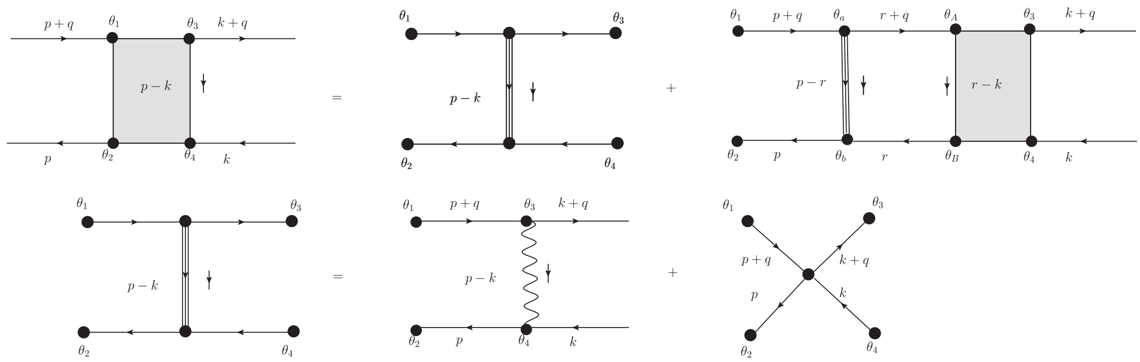

Figure 4. The diagrams in the first line pictorially represents the Schwinger-Dyson equation for offshell four point function (see (3.29)). The second line represents the tree level contributions from the gauge superfield interaction and the quartic interactions.

As in the case of two point functions, it is easily demonstrated that the invariance of this action under supersymmetry constraints the coefficient function V that appears in (3.27) to obey the equation (3.24). As we have already explained above, the most general solution to this equation is given in equation (3.26) for a general shift invariant function F.

3.5 An integral equation for the offshell four point function

The coefficient function V of the quartic term of the exact IPI effective action may be shown to obey the integral equation (see figure 4 for a diagrammatic representation of this equation)

$$\begin{aligned} V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) &= V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) \\ &\quad + \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(N V_0(\theta_1,\theta_2,\theta_a,\theta_b,p,q,r) \\ &\quad\quad P(r+q,\theta_a,\theta_A) P(r,\theta_B,\theta_b) V(\theta_A,\theta_B,\theta_3,\theta_4,r,q,k) \Big). \end{aligned} \tag{3.29}$$

In (3.29) V_0 is the tree level contribution to V. V_0 receives contributions from the two diagrams depicted in figure 4. The explicit evaluation of V_0 is a straightforward exercise and we find (see appendix H.2 for details)

$$\begin{aligned} V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) &= \exp\left(\frac{1}{4} X.(p.X_{12} + q.X_{13} + k.X_{43})\right) \\ &\quad \Big(-\frac{i\pi w}{\kappa} X^-_{12} X^+_{12} X^-_{13} X^+_{13} X^-_{43} X^+_{43} \\ &\quad\quad - \frac{4\pi i}{\kappa (p-k)_{--}} X^+_{12} X^+_{13} X^+_{43} (X^-_{12} + X^-_{34}) \Big). \end{aligned} \tag{3.30}$$

In the above, the first term in the bracket is the delta function from the quartic interaction, the second term is from the tree diagram due to the gauge superfield exchange computed in section H.2.

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We now turn to the evaluation of the coefficient V in the exact 1PI effective action. There are 2^6 linearly independent functions of the six independent shift invariant Grassman variables $X^{\pm}_{12}$, $X^{\pm}_{13}$ and $X^{\pm}_{43}$. Consequently the most general V consistent with supersymmetry is parameterized by 64 unknown functions of the three independent momenta. V (and so F) is necessarily an even function of these variables. It follows that the most general function F can be parameterized in terms of 32 bosonic functions of p, k and q. In principle one could insert the most general supersymmetric F into the integral equation (3.29) and equate equal powers of θ_i on the two sides of (3.29) to obtain 32 coupled integral equations for the 32 unknown complex valued functions. One could, then, attempt to solve this system of equations. This procedure would obviously be very complicated and difficult to implement in practice. Focusing on the special kinematics $q^{\pm} = 0$ we were able to shortcircuit this laborious process, in a manner we now describe.

After a little playing around we were able to demonstrate that V of the form[25]

$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = \exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right)F(X_{12},X_{13},X_{43},p,q,k)$$
$$F(X_{12},X_{13},X_{43},p,q,k) = X^+_{12}X^+_{43}\Big(A(p,k,q)X^-_{12}X^-_{43}X^+_{13}X^-_{13}+B(p,k,q)X^-_{12}X^-_{43}$$
$$+C(p,k,q)X^-_{12}X^+_{13}+D(p,k,q)X^+_{13}X^-_{43}\Big)\,, \tag{3.31}$$

is closed under the multiplication rule induced by the r.h.s. of (3.29) (see appendix H.3). Plugging in the general form of V (3.31) in the integral equation (3.29) and performing the grassmann integration, (3.29) turns into to the following integral equations for the coefficient functions A, B, C and D:

$$\begin{aligned}&A(p,k,q)+\frac{2\pi iw}{\kappa}\\&\quad+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2A(q_3p_-+2(q_3-im)r_-)+(q_3r_-+2imp_-)(2Bq_3+Ck_-)-Dr_-(q_3p_-+2imr_-)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-}\\&\quad-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{4iAm+2Bq_3^2+Cq_3k_-+2D(q_3+im)r_-}{(r^2+m^2)((r+q)^2+m^2)}=0\end{aligned} \tag{3.32}$$

$$\begin{aligned}&B(p,k,q)+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2A(p+r)_-+4B(q_3r_-+im(p-r)_-)-Ck_-(p+r)_--Dr_-(p-3r)_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-}\\&\quad-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{2A+4imB-Ck_--Dr_-}{(r^2+m^2)((r+q)^2+m^2)}=0\end{aligned} \tag{3.33}$$

$$\begin{aligned}&C(p,k,q)-\frac{4\pi i}{\kappa(p-k)_-}+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{2C\big(q_3(p+3r)_-+2im(p-r)_-\big)}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-}\\&\quad-i\pi\lambda w\int\frac{d^3r_E}{(2\pi)^3}\frac{2C(q_3+2im)}{(r^2+m^2)((r+q)^2+m^2)}=0\end{aligned} \tag{3.34}$$

$$\begin{aligned}&D(p,k,q)-\frac{4\pi i}{\kappa(p-k)_-}\\&\quad+i\pi\lambda\int\frac{d^3r_E}{(2\pi)^3}\frac{-A(4q_3-8im)+(q_3-2im)(4Bq_3+2Ck_-)+2D(3q_3+2im)r_-}{(r^2+m^2)((r+q)^2+m^2)(p-r)_-}=0\,.\end{aligned} \tag{3.35}$$

[25]The variables X, X_{ij} are defined in terms of θ_i in (3.25).

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We will sometimes find it useful to view the four integral equations above as a single integral equation for a four dimensional column vector E whose components are the functions A, B, C, D, i.e.

$$E(p,k,q) = \begin{pmatrix} A(p,k,q) \\ B(p,k,q) \\ C(p,k,q) \\ D(p,k,q) \end{pmatrix}. \tag{3.36}$$

The integral equations take the schematic form

$$E = R + IE \tag{3.37}$$

where R is a 4 column of functions and I is a matrix of integral operators acting on E. The integral equation (3.37) may be converted into a differential equation by differentiating both sides of (3.37) w.r.t. p_+. Using (H.36) and performing all d^3r integrals (using (H.34) for the integral over r_3) we obtain the differential equations

$$\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q)\,, \tag{3.38}$$

where

$$S(p,k,q) = -\frac{8i\pi^2}{\kappa}\delta^2((p-k)_-,(p-k)_+)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{3.39}$$

$$H(p,k_-,q_3) = \frac{1}{a(p_s,q_3)}\begin{pmatrix} (6q_3-4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2 \\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2 \\ 0 & 0 & 8q_3p_- & 0 \\ 8im-4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_- \end{pmatrix} \tag{3.40}$$

and

$$a(p_s,q_3) = \frac{\sqrt{m^2+p_s^2}\,(4m^2+q_3^2+4p_s^2)}{2\pi}\,. \tag{3.41}$$

As we have explained above, the exact vertex V enjoys invariance under the $\mathbb{Z}_2$ transformation (3.28). In terms of the functions A, B, C, D, the $\mathbb{Z}_2$ action is given by

$$E(p,k,q) = TE(k,p,-q)\,, \tag{3.42}$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.43}$$

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The differential equations (3.38) do not manifestly respect the invariance (3.42). In fact in appendix H.4 we have demonstrated that the differential equations (3.38) admit solutions that enjoy the invariance (3.42) if and only if the following consistency condition is obeyed:

$$[H(p,k_-,q), TH(k,p_-,-q)T] = 0\,. \tag{3.44}$$

In the same appendix we have also explicitly verified that this integrability condition is in fact obeyed; this is a consistency check on (3.38) and indirectly on the underlying integral equations.

3.6 Explicit solution for the offshell four point function

In this subsection, we solve the system of integral equations for the unknown functions A, B, C, D presented in the previous subsection. We propose the ansatz

$$\begin{aligned}
A(p,k,q) &= A_1(p_s,k_s,q_3) + \frac{A_2(p_s,k_s,q_3)k_-}{(p-k)_-}\,,\\
B(p,k,q) &= B_1(p_s,k_s,q_3) + \frac{B_2(p_s,k_s,q_3)k_-}{(p-k)_-}\,,\\
C(p,k,q) &= -\frac{C_2(p_s,k_s,q_3) - C_1(p_s,k_s,q_3)k_+p_-}{(p-k)_-}\,,\\
D(p,k,q) &= -\frac{D_2(p_s,k_s,q_3) - D_1(p_s,k_s,q_3)k_-p_+}{(p-k)_-}\,.
\end{aligned} \tag{3.45}$$

Our ansatz (3.45)[26] fixes the solution in terms of 8 unknown functions of p_s, k_s and q_3.

Plugging the ansatz (3.45) into the integral equations (3.32)–(3.35), one can do the angle and r_3 integrals (using the formulae (H.35) and (H.34) respectively) leaving only the r_s integral to be performed. Differentiating this expression w.r.t. to p_s turns out to kill the r_s integral yielding differential equations in p_s for the eight equations above.[27] The resulting differential equations turn out to be exactly solvable. Assuming that the solution respects the symmetry (3.42), it turns out to be given in terms of two unknown functions of k_s and q_3. These can be thought of as the integration constants that are not fixed by the symmetry requirement (3.42). Plugging the solutions back into the integral equations we were able to determine these two integration functions of k_s and q_3 completely. We now report our results.

[26]We were able to arrive at this ansatz by first explicitly computing the one loop answer and observing the functional forms. Moreover, in previous work a very similar ansatz was already used to solve the integral equations for the fermions (see appendix F of [36]).

[27]Another way to obtain these differential equations is to plug the ansatz (3.45) directly into the differential equations (3.38).

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The solutions for A and B are

$$
\begin{aligned}
A_1(p_s,k_s,q_3) &= e^{-2i\lambda\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}}\left(G_1(k_s,q_3)\vphantom{\frac{1}{1}}\right.\\
&\qquad\left.+\frac{2\pi(w-1)(2m-iq_3)e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa\left(e^{\frac{i\pi\lambda q_3}{|q_3|}}(q_3(w+3)-2im(w-1))+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}\right)}\right),\\
A_2(p_s,k_s,q_3) &= e^{-2i\lambda\tan^{-1}\left(\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}G_2(k_s,q_3)\,,\\
B_1(p_s,k_s,q_3) &= \\
&\frac{2\pi A_1(p_s,k_s,q_3)}{q_3}\\
&+\frac{2\pi}{b_1b_2}\left(-i(w-1)^2(4m^2+q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+4\tan^{-1}\frac{2|m|}{q_3}\right)}\right.\\
&+i(w-1)^2(-4m^2+8imq_3+3q_3^2)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}\right)}\\
&-8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)\right)}\\
&+(w-1)(q_3+2im)(2m(w-1)+iq_3(w+3))+e^{2i\lambda\left(\frac{\pi q_3}{|q_3|}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)}\\
&\left.+(w-1)(2m-3iq_3)(q_3(w+3)+2im(w-1))+e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)}\right),\\
B_2(p_s,k_s,q_3) &= \frac{A_2(p_s,k_s,q_3)}{q_3}\,,\\
G_1(k_s,q_3) &= -\frac{2\pi}{\kappa}\frac{1}{g_1}\left(-8iq_3^2(w+1)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}+\tan^{-1}\frac{2|m|}{q_3}\right)\right)}\right.\\
&\quad+i(w-1)^2(q_3-2im)^2e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+4\tan^{-1}\frac{2|m|}{q_3}\right)}\\
&\quad\left.-(w-1)(q_3-2im)(2m(w-1)+iq_3(w+3))e^{2i\lambda\left(\frac{\pi q_3}{|q_3|}+\tan^{-1}\frac{2|m|}{q_3}\right)}\right),\\
G_2(k_s,q_3) &= 0\,, \qquad (3.46)
\end{aligned}
$$

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where we have defined some parameters as given below for ease of presentation.

$$
\begin{aligned}
g_1 &= (w-1)(q_3+2im)e^{\frac{2i\pi\lambda q_3}{|q_3|}}\left(q_3(w+3)-2im(w-1)\right), \\
&\quad + (w-1)(4m^2(w-1)-8imq_3+q_3^2(w+3))e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}}, \\
&\quad - 2(4m^2(w-1)^2+q_3^2(w^2+2w+5))e^{i\lambda\left(\frac{\pi q_3}{|q_3|}+2\tan^{-1}\frac{2|m|}{q_3}\right)}, \\
b_1 &= \kappa q_3\left((w-1)(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+(-q_3(w+3)-2im(w-1))e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}\right), \\
b_2 &= e^{\frac{i\pi\lambda q_3}{|q_3|}}\left(q_3(w+3)-2im(w-1)\right)+i(w-1)(2m+iq_3)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}},
\end{aligned}
\tag{3.47}
$$

The solutions for C and D are

$$
\begin{aligned}
&C_1(p_s,k_s,q_3)= \\
&\frac{4\pi(q_3+2im)\left(e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}-e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}}\right)e^{i\lambda\left(\frac{\pi q_3}{|q_3|}-2\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa k_s^2\left(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+\left(2m-iq_3\left(\frac{w+3}{w-1}\right)\right)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}\right)}, \\
&C_2(p_s,k_s,q_3)= \\
&\frac{4\pi e^{2i\lambda\left(\tan^{-1}\frac{2|m|}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}\left((q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}-\left(q_3\left(\frac{w+3}{w-1}\right)+2im\right)e^{2i\lambda\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}}\right)}{\kappa\left(i(q_3+2im)e^{\frac{i\pi\lambda q_3}{|q_3|}}+\left(2m-iq_3\left(\frac{w+3}{w-1}\right)\right)e^{2i\lambda\tan^{-1}\frac{2|m|}{q_3}}\right)}, \\
&\qquad D_1(p_s,k_s,q_3)=C_1(k_s,p_s,-q_3), \\
&\qquad D_2(p_s,k_s,q_3)=C_2(k_s,p_s,-q_3).
\end{aligned}
\tag{3.48}
$$

It is straightforward to show that the above solutions satisfy the various symmetry requirements that follow from (3.42).

Although the solutions (3.46) and (3.48) are quite complicated, a drastic simplification occurs at the $\mathcal{N}=2$ point $w=1$

$$
\begin{aligned}
A &= -\frac{2i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa}, \\
B &= 0, \\
C &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-}, \\
D &= -\frac{4i\pi e^{2i\lambda\left(\tan^{-1}\frac{2\sqrt{k_s^2+m^2}}{q_3}-\tan^{-1}\frac{2\sqrt{m^2+p_s^2}}{q_3}\right)}}{\kappa(k-p)_-}.
\end{aligned}
\tag{3.49}
$$

It is satisfying that the complicated results of the general $\mathcal{N}=1$ theory collapse to an extremely simple form at the $\mathcal{N}=2$ point.

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3.7 Onshell limit and the S matrix

The explicit solution for the functions A, B, C and D, presented in the previous subsection, completely determine V in (3.27), and so the quadratic part of the exact (large N) IPI effective action. The most general 2×2 S matrix may now be obtained from (3.27) as follows. We simply substitute the onshell expressions

$$\begin{aligned}\Phi(p,\theta) = (2\pi)\delta(p^2+m^2)\Big[\theta(p^0)\Big(a(\mathbf{p})(1+m\theta^2)+\theta^\alpha u_\alpha(\mathbf{p})\alpha(\mathbf{p})\Big)\\ +\theta(-p^0)\Big(a^{c\dagger}(-\mathbf{p})(1+m\theta^2)+\theta^\alpha v_\alpha(-\mathbf{p})\alpha^{c\dagger}(-\mathbf{p})\Big)\Big]\end{aligned} \tag{3.50}$$

into (3.27) (here a and α are the effectively free oscillators that create and destroy particles at very early or very late times; these oscillators obey the commutation relations (2.29)). Performing the integrals over θ^α reduces (3.27) to a quartic form (let us call it L) in bosonic and fermionic oscillators. The S matrix is obtained by sandwiching the resultant expression between the appropriate in and out states, and evaluating the resulting matrix elements using the commutation relations (2.29).

It may be verified that the quartic form in oscillators takes the form[28]

$$\begin{aligned}L = \sum_{\phi_i=0,\pi}\int\prod_{i=1}^{4} d\theta_i\frac{d^3p_i}{((2\pi)^3)^4}\delta(p_i^2+m^2)S_M(p_1,\phi_1,\theta_1,p_2,\phi_2,\theta_2,p_3,\phi_3,\theta_3,p_4,\phi_4,\theta_4)\\ \Big(\delta_{\phi_i,0}\theta(p_i^0)A(p_i,\phi_i,\theta_i)+\delta_{\phi_i,\pi}\theta(-p_i^0)\tilde{A}(-p_i,\phi_i,\theta_i)\Big)(2\pi)^3\delta^3(p_1+p_2+p_3+p_4)\end{aligned}$$

where

$$\begin{aligned}A(p_i,\phi_i,\theta_i) &= a(\mathbf{p}_i)+\alpha(\mathbf{p}_i)e^{-\frac{i\phi_i}{2}}\theta_i\,,\\ \tilde{A}(p_i,\phi_i,\theta_i) &= a^\dagger(\mathbf{p}_i)+e^{-\frac{i\phi_i}{2}}\theta_i\alpha^\dagger(\mathbf{p}_i)\,,\end{aligned} \tag{3.51}$$

where the one component fermionic variables θ_i are the fermionic variables that parameterize onshell superspace (see section 2.4) and the master formula is defined in (2.50). Note that the phase variables ϕ_i are summed over two values 0 and π; the symbol $\delta_{\phi,0}$ is unity when $\phi = 0$ but zero when $\phi = \pi$ and $\delta_{\phi,\pi}$ has an analogous definition. (3.51) compactly identifies the coefficient of every quartic form in oscillators. For instance it asserts that the coefficient of $a_1a_2a_3^\dagger a_4^\dagger$ is the S matrix for scattering bosons with momentum p_1, p_2 to bosons with momenta p_3, p_4, while the the coefficient of $\alpha_2\alpha_4a_1^\dagger a_3^\dagger$ is minus the S matrix for scattering fermions with momentum p_2, p_4 to bosons with momentum p_1, p_3, etc.

We can use the δ function in (3.51) to perform the integral over one of the four momenta; the integral over the remaining momenta may be recast as an integral over the momenta p k and q employed in the previous section; specifically (see figure 4)

$$p_1 = p+q\,,\ p_2 = -k-q\,,\ p_3 = -p\,,\ p_4 = k\,. \tag{3.52}$$

From the explicit results we get by substituting (3.50) into (3.27) we can read off all S matrices at $q_\pm = 0$.

[28]The definition of A and $\tilde{A}$ reduces to the definition (2.32) for $\phi = 0$. While for $\phi = \pi$, it reduces to (2.32) together with the identification $\theta \to i\theta$. With these definitions $\tilde{A} = A^\dagger$ both at $\phi = 0, \pi$.

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To start with, let us restrict our attention to the bosonic sector. From direct computation[29] we find that in this sector (3.51) reduces to

$$
\begin{aligned}
L_B = \sum_{\phi_i=0,\pi} \int \frac{d^3p}{(2\pi)^3}\frac{dq_3}{(2\pi)}\frac{d^3k}{(2\pi)^3}&\delta((p+q)^2+m^2)\delta((k+q)^2+m^2)\\
&\delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_B(p,k,q_3)\\
&\Big(\delta_{\phi_i,0}\theta(p^0)a(\mathbf{p}+\mathbf{q})+\delta_{\phi_i,\pi}\theta(-p^0)a^\dagger(-\mathbf{p}-\mathbf{q})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(-k^0)a(-\mathbf{k}-\mathbf{q})+\delta_{\phi_i,\pi}\theta(k^0)a^\dagger(\mathbf{k}+\mathbf{q})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(-p^0)a(-\mathbf{p})+\delta_{\phi_i,\pi}\theta(p^0)a^\dagger(\mathbf{p})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(k^0)a(\mathbf{k})+\delta_{\phi_i,\pi}\theta(-k^0)a^\dagger(-\mathbf{k})\Big)
\end{aligned}
\tag{3.54}
$$

while for the purely fermionic sector (3.51) reduces to

$$
\begin{aligned}
L_F = \sum_{\phi_i=0,\pi} \int \frac{d^3p}{(2\pi)^3}\frac{dq_3}{(2\pi)}\frac{d^3k}{(2\pi)^3}&\delta((p+q)^2+m^2)\delta((k+q)^2+m^2)\\
&\delta(p^2+m^2)\delta(k^2+m^2)\mathcal{T}_F(p,k,q_3)\\
&\Big(\delta_{\phi_i,0}\theta(p^0)\alpha(\mathbf{p}+\mathbf{q})+\delta_{\phi_i,\pi}\theta(-p^0)\alpha^\dagger(-\mathbf{p}-\mathbf{q})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(-k^0)\alpha(-\mathbf{k}-\mathbf{q})+\delta_{\phi_i,\pi}\theta(k^0)\alpha^\dagger(\mathbf{k}+\mathbf{q})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(-p^0)\alpha(-\mathbf{p})+\delta_{\phi_i,\pi}\theta(p^0)\alpha^\dagger(\mathbf{p})\Big)\\
&\Big(\delta_{\phi_i,0}\theta(k^0)\alpha(\mathbf{k})+\delta_{\phi_i,\pi}\theta(-k^0)\alpha^\dagger(-\mathbf{k})\Big)
\end{aligned}
\tag{3.55}
$$

where[30]

$$
\mathcal{T}_B = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2}+J_B(q,\lambda)\,, \tag{3.56}
$$

$$
\mathcal{T}_F = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2}+J_F(q,\lambda)\ , \tag{3.57}
$$

[29]Note that the onshell delta functions in the equations (3.54) and (3.55) ensure that

$$
p_3=k_3=-\frac{q_3}{2}\ ,\ p_s=k_s\ ,\ \ k_s=\frac{i}{2}\sqrt{q_3^2+4m^2}\,. \tag{3.53}
$$

[30]Our actual computations gave the functions J_B and J_F in the special case $q^\pm=0$. We obtained the answers reported in (3.56) and (3.57) by determining the unique covariant expression that reduce to our answers for our special kinematics. While this procedure is completely correct (with standard conventions) for J_B, it is a bit inaccurate for J_F. The reason for this is that J_F is Lorentz invariant only upto a phase. As we have explained around (2.53), the phase of J_F depends on the (arbitrary) phase of the u and v spinors of the particles in the scattering process. The accurate answer is obtained by covariantizing the unambiguous $\mathcal{S}_f$ defined in (2.54). $\mathcal{S}_F$ is obtained by multiplying this result by the quadrilinear term in spinor wavefunctions as defined in (2.65). This gives an explicit but cumbersome expression for $\mathcal{S}_F$, which agrees with the result presented above upto an overall convention dependent phase. This phase vanishes near identity scattering (where it could have interfered with identity), and we have dealt with this issue carefully in deriving the unitarity equation. In the equation above we have simply ignored the phase in order to aid readability of formulas.

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where the J functions[31] are

$$J_B(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2}\,,$$
$$J_F(q,\lambda) = \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2}\,, \tag{3.58}$$

where

$$N_1 = \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)+(w-1)(2m-iq)\right)\,,$$
$$N_2 = \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(q(w+3)+2im(w-1))+(q(w+3)-2im(w-1))\right)\,,$$
$$M_1 = -8mq((w+3)(w-1)-4w)\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\,,$$
$$M_2 = -8mq(1+w)^2\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\,,$$
$$D_1 = \left(i\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)-2im(w-1)+q(w+3)\right)\,,$$
$$D_2 = \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(-q(w+3)-2im(w-1))+(w-1)(q+2im)\right)\,. \tag{3.59}$$

The equations (3.56) and (3.57) capture purely bosonic and purely fermionic S matrices in all channels (particle-particle scattering in the symmetric and antisymmetric channels as well as particle-antiparticle scattering in the adjoint channel) restricted to the kinematics $q_\pm = 0$. Recall that supersymmetry (see section 2.4) determines all other scattering amplitudes in terms of the four boson and four fermion amplitudes, so the formulae (3.56) and (3.57) are sufficient to determine all $2 \to 2$ scattering processes restricted to our special kinematics. In other words S_M in (3.51) is completely determined by (3.56) and (3.57) together with (2.50).

3.8 Duality of the S matrix

Under the duality transformation (see (2.12))

$$w' = \frac{3-w}{w+1}, \lambda' = \lambda - \mathrm{sgn}(\lambda), m' = -m, \kappa' = -\kappa \tag{3.60}$$

we have verified that

$$J_B(q,\kappa',\lambda',w',m') = -J_F(q,\kappa,\lambda,w,m)\,,$$
$$J_F(q,\kappa',\lambda',w',m') = -J_B(q,\kappa,\lambda,w,m)\,. \tag{3.61}$$

provided (2.15) is respected. In other words duality maps the purely bosonic and purely fermionic S matrices into one another. It follows that (3.56) and (3.57) map to each other

[31] The J functions are quite complicated and can be written in many avatars. In this section we have written the most elegant form of the J function, the other forms are reported in appendix I.

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under duality upto a phase. As we have explained in subsection 2.5, this result is sufficient to guarantee that the full S matrix (including, for instance, the S matrix for Bose-Fermi scattering) is invariant under duality, once we interchange bosons with fermions.

3.9 S matrices in various channels

In this subsection we explicitly list the purely bosonic and purely fermionic S matrices in every channel, as functions of the Mandelstam variables of that channel. These results are, of course, easily extracted from (3.54) and (3.55). There is a slight subtlety here; even though (3.56) and (3.57) are manifestly Lorentz invariant, it is not possible to write them entirely in terms of Mandelstam variables.[32] This is because (as was noted in [36]) $2+1$ dimensional kinematics allows for an additional Z_2 valued invariant (in addition to the Mandelstam variables)[33]

$$E(q, p-k, p+k) = \mathrm{Sign}\left(\epsilon_{\mu\nu\rho} q^\mu (p-k)^\nu (p+k)^\rho\right). \tag{3.63}$$

The sign of the first term in (3.56) and (3.57) is given by this new invariant as we will see in more detail below.

3.9.1 U channel

For particle-particle scattering

$$P_i(p_1) + P_j(p_2) \rightarrow P_i(p_3) + P_j(p_4)$$

we have the direct scattering referred to as the U_d (Symmetric) channel.[34] Our momenta assignments (see l.h.s. of figure 4) are

$$p_1 = p+q\,,\ p_2 = k\,,\ p_3 = p\,,\ p_4 = k+q\,. \tag{3.64}$$

In terms of the Mandelstam variables

$$s = -(p+q+k)^2\,,\ t = -q^2\,,\ u = -(p-k)^2\,, \tag{3.65}$$

the U_d channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned} \mathcal{T}_B^{U_d} &= E(q, p-k, p+k) \frac{4\pi i}{k} \sqrt{\frac{ts}{u}} + J_B(\sqrt{-t}, \lambda)\,, \\ \mathcal{T}_F^{U_d} &= E(q, p-k, p+k) \frac{4\pi i}{k} \sqrt{\frac{ts}{u}} + J_F(\sqrt{-t}, \lambda)\,. \end{aligned} \tag{3.66}$$

[32] We define the Mandelstam variables as usual

$$s = -(p_1+p_2)^2\,,\ t = -(p_1-p_3)^2\,,\ u = -(p_1-p_4)^2\,. \tag{3.62}$$

[33] Note, in particular that the expression (3.63) changes sign under the interchange of any two vectors.

[34] We adopt the terminology of [36] in specifying scattering channels; we refer the reader to that paper for a more complete definition of the U_d, U_e, T, and S channels that we will repeatedly refer to below.

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For the exchange scattering, referred to as the U_e (Antisymmetric) channel the momenta assignments are (see l.h.s. of figure 4)

$$p_1 = k\,,\ p_2 = p + q\,,\ p_3 = p\,,\ p_4 = k + q\,. \tag{3.67}$$

In terms of the Mandelstam variables

$$s = -(p + q + k)^2\,,\ t = -(p - k)^2\,,\ u = -q^2\,, \tag{3.68}$$

the U_e channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned}\mathcal{T}_B^{U_e} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + J_B(\sqrt{-u}, \lambda)\,,\\ \mathcal{T}_F^{U_e} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + J_F(\sqrt{-u}, \lambda)\,.\end{aligned} \tag{3.69}$$

3.9.2 T channel

For particle-antiparticle scattering

$$P_i(p_1) + A^j(p_2) \to P_i(p_3) + A^j(p_4)$$

S matrix in the adjoint channel is referred to as the T channel. The momentum assignments are (see l.h.s. of figure 4)

$$p_1 = p + q\,,\ p_2 = -k - q\,,\ p_3 = p\,,\ p_4 = -k\,. \tag{3.70}$$

In terms of the Mandelstam variables

$$s = -(p - k)^2\,,\ t = -q^2\,,\ u = -(p + q + k)^2\,, \tag{3.71}$$

the T channel T matrices for the boson-boson and fermion-fermion scattering are

$$\begin{aligned}\mathcal{T}_B^{T} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + J_B(\sqrt{-t}, \lambda)\,,\\ \mathcal{T}_F^{T} &= E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + J_F(\sqrt{-t}, \lambda)\,.\end{aligned} \tag{3.72}$$

In particle-antiparticle scattering there is also the singlet channel that we describe below.

3.10 The singlet (S) channel

We now turn to the most interesting scattering process; the scattering of particles with antiparticles in the S (singlet) channel. In this channel the external lines on the l.h.s. of figure 4 are assigned positive energy (and so represent initial states) while those on the right of the diagram are assigned negative energy (and so represent final states). It follows that we must make the identifications

$$p_1 = p + q\,,\ p_2 = -p\,,\ p_3 = k + q\,, p_4 = -k\,, \tag{3.73}$$

so that the Mandelstam variables for this scattering process are

$$s = -q^2\,,\ t = -(p - k)^2\,,\ u = -(p + k)^2\,. \tag{3.74}$$

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Note, in particular, that $s = -q^2$, and so is always negative when $q^\pm = 0$. As we have been able to evaluate the offshell correlator V (see (3.31)) only for $q^\pm = 0$, it follows that we cannot specialize our offshell computation to an onshell scattering process in the S channel in which $s \geq 4m^2$. In other words we do not have a direct computation of S channel scattering in any frame.

It is nonetheless tempting to simply assume that (3.56) and (3.57) continue to apply at every value of q^μ and not just when $q^\pm = 0$; indeed this is what the usual assumptions of analyticity of S matrices (and crossing symmetry in particular) would inevitably imply. Provisionally proceeding with this 'naive' assumption, it follows upon performing the appropriate analytic continuation ($q^2 \to -s$ for positive s; see section 4.4 of [36]) that

$$\begin{aligned} \mathcal{T}_B^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i\lambda \sqrt{\frac{su}{t}} + J_B(\sqrt{s}, \lambda)\,, \\ \mathcal{T}_F^{S;\text{naive}} &= E(q, p-k, p+k) 4\pi i\lambda \sqrt{\frac{su}{t}} + J_F(\sqrt{s}, \lambda)\,, \end{aligned} \tag{3.75}$$

where

$$\begin{aligned} J_B(\sqrt{s}, \lambda) &= -4\pi i\lambda\sqrt{s}\frac{N_1 N_2 + M_1}{D_1 D_2}\,, \\ J_F(\sqrt{s}, \lambda) &= -4\pi i\lambda\sqrt{s}\frac{N_1 N_2 + M_2}{D_1 D_2}\,, \end{aligned} \tag{3.76}$$

where

$$\begin{aligned} N_1 &= \left((w-1)(2m+\sqrt{s}) + (w-1)(2m-\sqrt{s})e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\right)\,, \\ N_2 &= \left((-i\sqrt{s}(w+3)+2im(w-1))+(-i\sqrt{s}(w+3)-2im(w-1))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\right), \\ M_1 &= 8mi\sqrt{s}((w+3)(w-1)-4w)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\,, \\ M_2 &= 8mi\sqrt{s}(1+w)^2 e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\,, \\ D_1 &= \left(i(w-1)(2m+\sqrt{s}) - (2im(w-1)+i\sqrt{s}(w+3))e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\right)\,, \\ D_2 &= \left((\sqrt{s}(w+3)-2im(w-1)) + (w-1)(-i\sqrt{s}+2im)e^{i\pi\lambda}\left(\frac{\sqrt{s}+2|m|}{\sqrt{s}-2|m|}\right)^\lambda\right). \end{aligned} \tag{3.77}$$

Including the identity factors, the naive S channel S matrix that follows from the usual rules of crossing symmetry are

$$\begin{aligned} \mathcal{S}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4), \\ \mathcal{S}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4), \end{aligned} \tag{3.78}$$

where the identity operator is defined in (2.57).

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We pause here to note a subtlety. The quantity $\mathcal{S}_F^{S;\text{naive}}$ quoted above equals the S matrix in the S channel only upto phase. In order to obtain the fully correct S matrix we analytically continue the phase unambiguous quantity $\mathcal{S}_f^{S;\text{naive}}$.[35] The result of that continuation is given by

$$\mathcal{S}_f^{S;\text{naive}} = \frac{\mathcal{S}_F^{S;\text{naive}}}{X(s)} \tag{3.79}$$

where[36]

$$X(s) = -\frac{-s+4m^2}{4m^2} = -4Y(s)\,. \tag{3.81}$$

The full four fermion amplitude in the S channel, including phase is then given by

$$A_F^{S;\text{naive}} = \mathcal{S}_f^{S;\text{naive}} X(p,k,q)$$

where[37]

$$X(p,k,q) = \frac{1}{4m^2}\left(u(p+q)u(-p)\right)\left(v(k+q)v(-k)\right)\,. \tag{3.82}$$

It is not difficult to check that

$$|X(p,k,q)| = X(s)\,.$$

It follows that the S channel 4 fermion amplitude agrees with $\mathcal{S}_F$ upto a convention dependent phase. This phase factor may be shown to vanish near the identity momentum configuration ($p_1 = p_3$, $p_2 = p_4$) and so does not affect the interference with identity, and in general has no physical effect; it follows we would make no error if we simply regarded $\mathcal{S}_F$ as the four fermion scattering amplitude. At any rate we have been careful to express the unitarity relation in terms of the phase unambiguous quantity $\mathcal{S}_f$ given unambiguously by (2.54).

The naive S channel S matrix (3.78) is not duality (2.12) invariant. In later section, we also show that it also does not obey the constraints of unitarity, leading to an apparent paradox.

A very similar paradox was encountered in [36] where it was conjectured that the usual rules of crossing symmetry are modified in matter Chern-Simons theories. It was conjectured in [36] that the correct transformation rule under crossing symmetry for *any* matter Chern-Simons theory with fundamental matter in the large N limit is given by

$$\begin{aligned}\mathcal{S}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,,\\ \mathcal{S}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,,\end{aligned} \tag{3.83}$$

[35] Indeed it does not make sense to analytically continue $\mathcal{S}_F$ as the ambiguous phases of this quantity are not necessarily Lorentz invariant, and so are not functions only of the Mandelstam variables.

[36] The factor of $X(s)$ is the analytic continuation of (see (2.54))

$$\left(\bar{u}(p)u(p+q)\right)\left(\bar{v}(-k-q)v(-k)\right) = X(q) = -\frac{q^2+4m^2}{4m^2}\,. \tag{3.80}$$

The analytic continuation of the above formula is same as $-4Y(s)$ (see (2.77)).

[37] The spinor quadrilinear is as defined in (2.65) with momentum assignments corresponding to the S channel (3.73).

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where

$$\mathcal{T}_B^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,,$$
$$\mathcal{T}_F^S(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) = -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,,\tag{3.84}$$

where (3.75) defines the T matrices obtained from naive crossing rules. In the center of mass frame the conjectured S matrix (3.83) has the form

$$\mathcal{S}_B^S(s,\theta) = 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta)\,,$$
$$\mathcal{S}_F^S(s,\theta) = 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta)\,,\tag{3.85}$$

where

$$\mathcal{T}_B^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive}}(s,\theta)\,,$$
$$\mathcal{T}_F^S(s,\theta) = -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive}}(s,\theta)\,.\tag{3.86}$$

The naive analytically continued T matrices are

$$\mathcal{T}_B^{S;\text{naive}}(s,\theta) = 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\,,$$
$$\mathcal{T}_F^{S;\text{naive}}(s,\theta) = 4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\,,\tag{3.87}$$

where the J functions are as defined in (3.76). In other words the conjectured S matrix takes the following form

$$\mathcal{S}_B^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right)\,,$$
$$\mathcal{S}_F^S(s,\theta) = 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right)\,.\tag{3.88}$$

It was demonstrated in [36] that the conjecture (3.84) yields an S channel S matrix that is both duality invariant and consistent with unitarity in the the systems under study in that paper. In this paper we will follow [36] to conjecture that (3.84) continues to define the correct S channel S matrix for the theories under study. In the next section we will demonstrate that (3.84) obeys the nonlinear unitarity equations (2.73) and (2.74). We regard this fact as highly nontrivial evidence in support of the conjecture (3.84). As (3.84) appears to work in at least two rather different classes of large N fundamental matter Chern-Simons theories (namely the purely bosonic and fermionic theories studied in [36] and the supersymmetric theories studied in this paper) it seems likely that (3.84) applies universally to all Chern-Simons fundamental matter theories, as suggested in [36].

3.10.1 Straightforward non-relativistic limit

The conjectured S channel S matrix has a simple non-relativistic limit leading to the known Aharonov-Bohm result (see section 2.6 of [36] for details). In this limit we take (in the

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center of mass frame) $\sqrt{s} \to 2m$ in the T matrix (3.86) with all other parameters held fixed. In this limit we find

$$
\begin{aligned}
\mathcal{T}_B^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)-1\right)\,, \\
\mathcal{T}_F^S(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + 4\sqrt{s}\sin(\pi\lambda)\left(i\cot(\theta/2)+1\right). \qquad (3.89)
\end{aligned}
$$

The non-relativistic limit also coincides with the $\mathcal{N}=2$ limit of the S matrix (3.83) as we show in the following subsection. In section 5.5 we describe a slightly modified non-relativistic limit of the S matrix.

3.11 S matrices in the $\mathcal{N}=2$ theory

As discussed in section 2.1 the $\mathcal{N}=1$ theory (2.1) has an enhanced $\mathcal{N}=2$ supersymmetric regime when the Φ^4 coupling constant takes a special value $w=1$. We have already seen that the momentum dependent functions in the offshell four point function simplify dramatically (3.49), and so it is natural to expect that the S matrices at $w=1$ are much simpler than at generic w. This is indeed the case as we now describe.

By taking the limit $w \to 1$ in the S matrix formulae presented in (3.56) and (3.57), we find that the four boson and four fermion $\mathcal{N}=2$ S matrices take the very simple form[38]

$$
\mathcal{T}_B^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} - \frac{8\pi m}{\kappa}\,, \qquad (3.90)
$$

$$
\mathcal{T}_F^{\mathcal{N}=2} = \frac{4i\pi}{\kappa}\epsilon_{\mu\nu\rho}\frac{q^\mu(p-k)^\nu(p+k)^\rho}{(p-k)^2} + \frac{8\pi m}{\kappa}\,. \qquad (3.91)
$$

The S matrices above are simply those for tree level scattering. It follows that the tree level S matrices in the three non-anyonic channels are not renormalized, at any order in the coupling constant, in the $\mathcal{N}=2$ theory.

There is an immediate (but rather trivial) check of this result. Recall that according to section C the four boson and four fermion scattering amplitudes are not independent in the $\mathcal{N}=2$ theory; supersymmetry determines the former in terms of the latter. The precise relation is derived in C and is given by (C.24) for particle-antiparticle scattering and (C.29) for particle-particle scattering. It is easy to verify that (3.90) and (3.91) trivially satisfy (C.24) (or (C.29)) using (2.38), (2.39) and appropriate momentum assignments for the channels of scattering discussed in section 3.9.[39]

For completeness we now present explicit formulae for the S matrices of the $\mathcal{N}=2$ theory in the three non-anyonic channels.

---

[38] This is because the J functions reported in (3.56) and (3.57) have an extremely simple form at $w=1$ (see (I.13)).

[39] As an example, in the T channel (see (3.70)) we substitute the coefficients (2.38), (2.39) into (C.24) and evaluate it to get

$$
\mathcal{S}_B = \mathcal{S}_F \frac{-2m(k-p)_- + iq_3(k+p)_-}{2m(k-p)_- + iq_3(k+p)_-}\,. \qquad (3.92)
$$

It is clear that the covariant form of the S matrices given in (3.90) and (3.91) trivially satisfy (3.92). Similarly it can be easily checked that the result (3.92) follows from (C.29) for particle-particle scattering.

For the U_d channel

$$\mathcal{T}_B^{U_d;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} - \frac{8\pi m}{\kappa},$$
$$\mathcal{T}_F^{U_d;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{ts}{u}} + \frac{8\pi m}{\kappa}. \tag{3.93}$$

For the U_e channel

$$\mathcal{T}_B^{U_e;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} - \frac{8\pi m}{\kappa},$$
$$\mathcal{T}_F^{U_e;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{us}{t}} + \frac{8\pi m}{\kappa}. \tag{3.94}$$

For the T channel

$$\mathcal{T}_B^{T;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} - \frac{8\pi m}{\kappa},$$
$$\mathcal{T}_F^{T;\mathcal{N}=2} = E(q, p-k, p+k)\frac{4\pi i}{k}\sqrt{\frac{tu}{s}} + \frac{8\pi m}{\kappa}. \tag{3.95}$$

Let us now turn to the singlet channel. As described in section 3.10, we cannot compute the S channel S matrix directly because of our choice of the kinematic regime $q_\pm = 0$. The naive analytic continuation of (3.90) and (3.91) to the S channel gives

$$\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2} = E(q, p-k, p+k)4\pi i\lambda\sqrt{\frac{su}{t}} - 8\pi m\lambda,$$
$$\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2} = E(q, p-k, p+k)4\pi i\lambda\sqrt{\frac{su}{t}} + 8\pi m\lambda. \tag{3.96}$$

Thus the naive S channel S matrix for the $\mathcal{N}=2$ theory is

$$\begin{aligned}\mathcal{S}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\\ &\quad + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4),\\ \mathcal{S}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\\ &\quad + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4).\end{aligned} \tag{3.97}$$

As explained in the introduction 1, this result is obviously non-unitary. Applying the modified crossing symmetry transformation rules (3.83) we obtain our conjecture for the $\mathcal{N}=2$ S matrix in the singlet channel

$$\begin{aligned}\mathcal{S}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4),\\ \mathcal{S}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) + i(2\pi)^3\delta^3(p_1+p_2-p_3-p_4)\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4),\end{aligned} \tag{3.98}$$

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where

$$\begin{aligned}
\mathcal{T}_B^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \\
&\quad + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_B^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,, \\
\mathcal{T}_F^{S;\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= -i(\cos(\pi\lambda)-1)I(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) \\
&\quad + \frac{\sin(\pi\lambda)}{\pi\lambda}\mathcal{T}_F^{S;\text{naive};\mathcal{N}=2}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\,.
\end{aligned} \tag{3.99}$$

In the center of mass frame the conjectured S channel S matrix in the $\mathcal{N}=2$ theory takes the form

$$\begin{aligned}
\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_B^S(s,\theta)\,, \\
\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\delta(\theta) + i\mathcal{T}_F^S(s.\theta)\,,
\end{aligned} \tag{3.100}$$

where

$$\begin{aligned}
\mathcal{T}_B^{S;\mathcal{N}=2}(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2)-8m)\,, \\
\mathcal{T}_F^{S;\mathcal{N}=2}(s,\theta) &= -8\pi i\sqrt{s}(\cos(\pi\lambda)-1)\delta(\theta) + \sin(\pi\lambda)(4i\sqrt{s}\cot(\theta/2)+8m)\,.
\end{aligned} \tag{3.101}$$

Note that as $\sqrt{s}\to 2m$ (3.101) reproduces the straightforward non-relativistic limit of the $\mathcal{N}=1$ theory (3.89).

In other words the conjectured S channel S matrix for the $\mathcal{N}=2$ theory takes the following form in the center of mass frame

$$\begin{aligned}
\mathcal{S}_B^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2)-8m\right)\,, \\
\mathcal{S}_F^{S;\mathcal{N}=2}(s,\theta) &= 8\pi\sqrt{s}\cos(\pi\lambda)\delta(\theta) + i\sin(\pi\lambda)\left(4i\sqrt{s}\cot(\theta/2)+8m\right)\,.
\end{aligned} \tag{3.102}$$

We explicitly show that the conjectured S channel S matrix is unitary in the following section.

4 Unitarity

In this section, we first show that the S matrices in the T and U channel obey the unitarity conditions (2.73) and (2.74) at leading order in the large N limit. As the relevant unitarity equations are linear, the unitarity equation is a relatively weak consistency check of the S matrices computed in this paper.

We then proceed to demonstrate that the S matrix (3.83) also obeys the constraints of unitarity. As the unitarity equation is nonlinear in the S channel, this constraint is highly nontrivial, we believe it provides an impressive consistency check of the conjecture (3.83).

4.1 Unitarity in the T and U channel

We begin by discussing the unitarity condition for the T (adjoint) and U (particle-particle) channels. Firstly we note that the S matrices in these channels are $O(1/N)$. Therefore

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the l.h.s. of (2.73) and (2.74) are $O(1/N^2)$. It follows that the unitarity equations (2.73) and (2.74) are obeyed at leading order in the large N limit provided

$$\begin{aligned}\mathcal{T}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= \mathcal{T}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)\,,\\ \mathcal{T}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4) &= \mathcal{T}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2)\,.\end{aligned} \tag{4.1}$$

The four boson and four fermion S matrices in the T channel are given in terms of the universal functions in (3.56) and (3.57) after applying the momentum assignments (3.70). It follows that (4.1) holds in the T channel provided

$$\begin{aligned}\mathcal{T}_B^T(p+q,-k-q,p,-k) &= \mathcal{T}_B^{T*}(p,-k,p+q,-k-q)\,,\\ \mathcal{T}_F^T(p+q,-k-q,p,-k) &= \mathcal{T}_F^{T*}(p,-k,p+q,-k-q)\,.\end{aligned} \tag{4.2}$$

This equation may be verified to be true (see below for some details).

Similarly the U_d channel S matrix is obtained via the momentum assignments (3.64); it follows that (4.1) is obeyed provided

$$\begin{aligned}\mathcal{T}_B^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_B^{U_d*}(p,k+q,p+q,k)\,,\\ \mathcal{T}_F^{U_d}(p+q,k,p,k+q) &= \mathcal{T}_F^{U_d*}(p,k+q,p+q,k)\,,\end{aligned} \tag{4.3}$$

which can also be checked to be true.

Finally in the U_e channel it follows from the momentum assignments (3.67) that (4.1) holds provided

$$\begin{aligned}\mathcal{T}_B^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_B^{U_e*}(p,k+q,k,p+q)\,,\\ \mathcal{T}_F^{U_e}(k,p+q,p,k+q) &= \mathcal{T}_F^{U_e*}(p,k+q,k,p+q)\,,\end{aligned} \tag{4.4}$$

which we have also verified.

The T matrices for all the above channels of scattering are reported in section 3.9. Note that the starring of the T matrices in (4.1) also involves a momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$. It follows that under this exchange $q \to -q$.[40]

In verifying (4.2), (4.3) and (4.4) we have used the fact that the functions J_B and J_F are both invariant under the combined operation of complex conjugation accompanied by the flip $q \to -q$ (see (I.9)). We also use the fact that in each case (T, U_d and U_e) the factor $E(q, p-k, p+k)$ flips sign under the momentum exchange $p_1 \Leftrightarrow p_3$ and $p_2 \Leftrightarrow p_4$; the sign obtained from this process compensates the minus sign from complex conjugating the explicit factor of i.[41]

[40]For instance in the T channel, we get the equations

$$p' + q' = p\,,\ \ p' = p + q\,,\ \ -k' - q' = -k\,,\ \ -k' = -k - q\,. \tag{4.5}$$

It follows that $q' = -q$.

[41]The unitarity conditions in these channels are simply the statement that the S matrices are real. The reality of S matrices is tightly connected to the absence of two particle branch cuts in the S matrices in these channels at leading order in large N.

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4.2 Unitarity in S channel

The S matrix in the S channel is of $O(1)$ and one has to use the full non-linear unitarity conditions (2.78) and (2.79). We reproduce them here for convenience.

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha))\,, \tag{4.6}$$
$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)+4Y(s)\mathcal{T}_f^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))+4Y(s)\mathcal{T}_f^{S*}(s,-(\alpha-\theta)))$$
$$-16Y(s)^2\mathcal{T}_f^S(s,\theta)\mathcal{T}_f^{S*}(s,-(\alpha-\theta))\Big)=i4Y(s)(-\mathcal{T}_f^S(s,\alpha)+\mathcal{T}_f^{S*}(s,-\alpha))\,, \tag{4.7}$$

where

$$Y(s)=\frac{-s+4m^2}{16m^2} \tag{4.8}$$

is as defined in (2.66), and $\mathcal{T}_B^S$ corresponds to the bosonic T matrix while $\mathcal{T}_f^S$ corresponds to the phase unambiguous part of the fermionic T matrix in the Singlet (S) channel given in (3.84) (also see (3.79)). In center of mass coordinates it takes the form

$$\mathcal{T}_f^S(s,\theta)=-\frac{\mathcal{T}_F^S(s,\theta)}{4Y(s)}\,. \tag{4.9}$$

Substituting the above into (4.6) and (4.7), the conditions for unitarity may be rewritten as

$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(-Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$+\mathcal{T}_B^S(s,\theta)\mathcal{T}_B^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_B^{S*}(s,-\alpha)-\mathcal{T}_B^S(s,\alpha))\,, \tag{4.10}$$
$$\frac{1}{8\pi\sqrt{s}}\int d\theta\Big(Y(s)(\mathcal{T}_B^S(s,\theta)-\mathcal{T}_F^S(s,\theta))(\mathcal{T}_B^{S*}(s,-(\alpha-\theta))-\mathcal{T}_F^{S*}(s,-(\alpha-\theta)))$$
$$-\mathcal{T}_F^S(s,\theta)\mathcal{T}_F^{S*}(s,-(\alpha-\theta))\Big)=i(\mathcal{T}_F^S(s,\alpha)-\mathcal{T}_F^{S*}(s,-\alpha))\,. \tag{4.11}$$

Let us pause to note that under duality $\mathcal{T}_B\to\mathcal{T}_F$ and vice versa; it follows then (4.10) and (4.11) map to each other under duality. In other words the unitarity conditions are compatible with duality.

We will now verify that our S channel S matrix is indeed compatible with unitarity. Let us recall that the angular dependence of the S matrix, in the center of mass frame is given by

$$\begin{aligned}\mathcal{T}_B^S&=H_BT(\theta)+W_B-iW_2\delta(\theta)\,,\\ \mathcal{T}_F^S&=H_FT(\theta)+W_F-iW_2\delta(\theta)\,,\end{aligned} \tag{4.12}$$

where

$$T(\theta)=i\cot(\theta/2).$$

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We will list the particular values of the coefficient functions $H_B(s)$ etc below; we will be able to proceed for a while leaving these functions unspecified.

Substituting (4.12) in (4.10) and doing the angle integrations[42] we find that (4.10) is obeyed if and only if

$$\begin{aligned}
&H_B - H_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_B^* - H_B W_2^*)\,, \\
&W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_B H_B^*)\,, \\
&W_B - W_B^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_B^* - W_2^* W_B) - \frac{i}{4\sqrt{s}}(H_B H_B^* - W_B W_B^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*).
\end{aligned} \tag{4.14}$$

Similarly (4.11) is obeyed if and only if

$$\begin{aligned}
&H_F - H_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 H_F^* - H_F W_2^*)\,, \\
&W_2 + W_2^* = -\frac{1}{8\pi\sqrt{s}}(W_2 W_2^* + 4\pi^2 H_F H_F^*)\,, \\
&W_F - W_F^* = \frac{1}{8\pi\sqrt{s}}(W_2 W_F^* - W_2^* W_F) - \frac{i}{4\sqrt{s}}(H_F H_F^* - W_F W_F^*) - \frac{iY}{4\sqrt{s}}(W_B - W_F)(W_B^* - W_F^*).
\end{aligned} \tag{4.15}$$

The first two equations of (4.14) and (4.15) are entirely identical to the first two equations of equation 2.66 in [36] for the non-supersymmetric case. The third equation has an additional contribution due to supersymmetry. Note that (4.14) and (4.15) are compatible with duality under $H_B \to H_F$ and $W_B \to W_F$ and vice versa.

Let us now proceed to verify that the equations (4.14) and (4.15) are indeed obeyed; for this purpose we need to use the specific values of the coefficient functions in (4.12). These functions are easily read off from the formulae (3.86) (that we reproduce here for convenience)

$$\begin{aligned}
\mathcal{T}_B^S &= -8\pi i\sqrt{s}(\cos(\pi\lambda) - 1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_B(\sqrt{s},\lambda)\right), \\
\mathcal{T}_F^S &= -8\pi i\sqrt{s}(\cos(\pi\lambda) - 1)\delta(\theta) + \frac{\sin(\pi\lambda)}{\pi\lambda}\left(4\pi i\lambda\sqrt{s}\cot(\theta/2) + J_F(\sqrt{s},\lambda)\right),
\end{aligned} \tag{4.16}$$

from which we find

$$\begin{aligned}
W_B &= J_B(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\,, \\
W_F &= J_F(\sqrt{s},\lambda)\frac{\sin(\pi\lambda)}{\pi\lambda}\,,
\end{aligned} \tag{4.17}$$

where the explicit form of the J functions are given in (3.76). While we also identify

$$H_B = H_F = 4\sqrt{s}\sin(\pi\lambda),\ \ W_2 = 8\pi\sqrt{s}(\cos(\pi\lambda) - 1),\ \ T(\theta) = i\cot(\theta/2)\,. \tag{4.18}$$

[42] The angle integrations in (4.10) can be done by using the formula

$$\int d\theta \mathrm{Pv}\cot\left(\frac{\theta}{2}\right)\mathrm{Pv}\cot\left(\frac{\alpha-\theta}{2}\right) = 2\pi - 4\pi^2\delta(\alpha), \tag{4.13}$$

where Pv stands for principal value. See (H.38) for a simple check of this formula.

Using the above relations it is very easy to see that the first two equations in each of (4.14) and (4.15) are satisfied. The first equation in each of (4.14) and (4.15) holds because H_B, H_F and W_2 are all real. The second equation in each case boils down to a true trigonometric identity.

The functions W_B and W_F occur only in the third equation in (4.14) and (4.15). These equations assert two nonlinear identities relating the (rather complicated) J_B and J_F functions. We have verified by explicit computation that these identities are indeed obeyed. It follows that the conjectured S matrix (3.83) is indeed unitary.

At the algebraic level, the satisfaction of the unitarity equation appears to be a minor miracle. A small mistake of any sort (a factor or two or an incorrect sign) causes this test to fail badly. In particular, unitarity is a very sensitive test of the conjectured form (3.83) of the S matrix. Let us recall again that this conjecture was first made in [36], where it was shown that it leads to a unitary $2 \to 2$ S matrix. The supersymmetric S matrices of this paper are more complicated than the S matrices of the purely bosonic or purely fermionic theories of [36]. In particular the unitarity equation for four boson and four fermion S matrices is different in this paper from the corresponding equations in [36] (the difference stems from the fact that two bosons can scatter not just to two bosons but also to two fermions, and this second process also contributes to the quadratic part of the unitarity equations). Nonetheless the prescription (3.83) adopted from [36] turns out to give results that obey the modified unitarity equation of this paper. In our opinion this constitutes a very nontrivial check of the crossing symmetry relation (3.83) proposed in [36].

The unitarity equation is satisfied for the arbitrary $\mathcal{N} = 1$ susy theory, and so is, in particular obeyed for the $\mathcal{N} = 2$ theory. Recall that the $\mathcal{N} = 2$ theory has a particularly simple S matrix (3.101). In fact in the T and U channels the $\mathcal{N} = 2$ S matrix is tree level exact at leading order in large N. According to the rules of naive crossing symmetry the S channel S matrix would also have been tree level exact. This result is in obvious conflict with the unitarity equation: in the equation $-i(T - T^\dagger) = TT^\dagger$ the l.h.s. vanishes at tree level while the r.h.s. is obviously nonzero. The modified crossing symmetry rules (3.83) resolve this paradox in a very beautiful way. According to the rules (3.83), the T matrix is not Hermitian even if T^{naive} is; as the term in (3.83) proportional to identity is imaginary. It follows from (3.83) that both l.h.s. and the r.h.s. of the unitarity equation are nonzero; they are infact equal, as we now pause to explicitly demonstrate. In the $\mathcal{N} = 2$ limit (see (3.101)) we have

$$\begin{aligned} H_B &= H_F = 4\sqrt{s}\sin(\pi\lambda)\,, \\ W_B &= -8m\sin(\pi\lambda)\,, \\ W_F &= 8m\sin(\pi\lambda)\,, \\ W_2 &= 8\pi\sqrt{s}(\cos(\pi\lambda) - 1)\,. \end{aligned} \tag{4.19}$$

The first equation in (4.14) is satisfied because everything is real. We have checked that the second equation is satisfied using a trigonometric identity.[43] The third equation works

[43]This is the only equation in which the l.h.s. and r.h.s. are both nonzero. The l.h.s. is the imaginary part of the coefficient of identity.

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because we have

$$(H_B H_B^* - W_B W_B^*) = -16\sin^2(\pi\lambda)(-s+4m^2) \tag{4.20}$$

and

$$Y(W_B - W_F)(W_B^* - W_F^*) = 16\sin^2(\pi\lambda)(-s+4m^2) \tag{4.21}$$

the other terms don't matter because everything else is real. The same thing is true for (4.15) since

$$(H_F H_F^* - W_F W_F^*) = -16\sin^2(\pi\lambda)(-s+4m^2) \tag{4.22}$$

and thus the unitarity conditions are satisfied by the conjectured S matrix (3.84) in the $\mathcal{N}=2$ theory as well.

5 Pole structure of S matrix in the S channel

The S channel S matrix studied in the last two sections turns out to have an interesting analytic structure. In this section we will demonstrate that the S matrix has a pole whenever $w < -1$. As we demonstrate below the pole is at threshold at $w = -1$, migrates to lower masses as w is further reduced until it actually occurs at zero mass at a critical value $w = w_c(\lambda) < -1$. As w is further reduced, the squared mass of the pole increases again, until the pole mass returns to threshold at $w = -\infty$.

In order to establish all these facts let us recall the structure of four boson and four fermion S matrix in the S channel. The S matrices take the form (see (3.76))

$$\mathcal{T}_B^S = \frac{n_b}{d_1 d_2}, \qquad \mathcal{T}_F^S = \frac{n_f}{d_1 d_2}, \tag{5.1}$$

where

$$\begin{aligned}
d_1 &= -4|m|^2\left(\text{sgn}(\lambda)(w-1)\left(\left(\frac{1+y}{1-y}\right)^{\lambda}-1\right)+y\left(-w\left(\frac{1+y}{1-y}\right)^{\lambda}+w+\left(\frac{1+y}{1-y}\right)^{\lambda}+3\right)\right),\\
d_2 &= \text{sgn}(\lambda)(w-1)\left(\left(\frac{1+y}{1-y}\right)^{\lambda}-1\right)+y\left(w\left(\left(\frac{1+y}{1-y}\right)^{\lambda}-1\right)+3\left(\frac{1+y}{1-y}\right)^{\lambda}+1\right),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
n_b &= -32|m|^3 y\sin(\pi\lambda)\left(8\ \text{sgn}(\lambda)(w+1)y\left(\frac{1+y}{1-y}\right)^{\lambda}\right.\\
&\quad+(w-1)(\text{sgn}(\lambda)-y)\left(\frac{1+y}{1-y}\right)^{2\lambda}(\text{sgn}(\lambda)(w-1)+(w+3)y)\\
&\quad\left.-(w-1)(\text{sgn}(\lambda)+y)(\text{sgn}(\lambda)(w-1)-(w+3)y)\right),\\
n_f &= 32|m|^3 y\sin(\pi\lambda)\left(8\ \text{sgn}(\lambda)(w+1)y\left(\frac{1+y}{1-y}\right)^{\lambda}\right.\\
&\quad-(w-1)(\text{sgn}(\lambda)-y)\left(\frac{1+y}{1-y}\right)^{2\lambda}(\text{sgn}(\lambda)(w-1)+(w+3)y)\\
&\quad\left.+(w-1)(\text{sgn}(\lambda)+y)(\text{sgn}(\lambda)(w-1)-(w+3)y)\right),
\end{aligned} \tag{5.3}$$

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where $y = \sqrt{s}/2|m|$. Through this discussion we assume that $\lambda m > 0$ (recall this condition was needed for duality invariance).

The denominators d_1, d_2 and the numerators are all polynomials of y and the quantity

$$X = \left(\frac{1+y}{1-y}\right)^{\lambda} .$$

Most of the interesting scaling behaviors we will encounter below are a consequence of the dependence of all quantities on X. Note that d_1 and d_2 are linear functions of X while n_b and n_f are quadratic functions of X. It is consequently possible to recast n_b and n_f in the form

$$\begin{aligned} n_b &= a_b d_1 d_2 + b_b d_1 + c_b d_2 \,, \\ n_f &= a_f d_1 d_2 + b_f d_1 + c_f d_2 \,. \end{aligned}$$

Here a_b, b_b, c_b, a_f, b_f and c_f are polynomials of y (but are independent of X) and are given by

$$\begin{aligned} a_b &= y \,, \\ b_b &= (w-1)(\mathrm{sgn}(\lambda)+y)^2 \,, \\ c_b &= -4|m|^2(\mathrm{sgn}(\lambda)-y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \,, \\ a_f &= y \,, \\ b_f &= -(w-1)\left(1-y^2\right) \,, \\ c_f &= 4|m|^2(\mathrm{sgn}(\lambda)+y)(\mathrm{sgn}(\lambda)(w-1)-(w+3)y) \,. \end{aligned} \tag{5.4}$$

In order to study the poles of the S matrix we need to investigate the zeroes of the functions d_1 and d_2. Let us first consider the case $\lambda > 0$. In this case it turns out that $d_1(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_2(y)$ has a zero in the range $w \in [w_c, -1]$ where

$$w_c(\lambda) = 1 - \frac{2}{|\lambda|} \,. \tag{5.5}$$

At $w = -\infty$ the zero of d_1 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_1 no longer has a zero. However $d_2(y)$ develops a zero. The zero of $d_2(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

When $\lambda < 0$ we have an identical situation except that the roles of d_1 and d_2 are reversed. $d_2(y)$ has a zero for $w \in (-\infty, w_c]$, while $d_1(y)$ has a zero in the range $w \in [w_c, -1]$. At $w = -\infty$ the zero of d_2 occurs at $y = 1$. As w is increased the y value of the zero decreases, until it reaches $y = 0$ at $w = w_c$. At larger values of w, d_2 no longer has a zero. However $d_1(y)$ develops a zero. The zero of $d_1(y)$ starts out at $y = 0$ when $w = w_c$, and then increases, reaching $y = 1$ at $w = -1$. At larger values of w neither d_1 nor d_2 have a zero.

In summary our S matrix has a pole for $w \in (-\infty, -1]$. The pole lies at threshold at the end points of this range, and becomes massless at $w = w_c$. There are clearly three special values of w in this range: $w = -1$, $w = w_c$ and $w = -\infty$. In the rest of this section we examine the neighborhood of three special points in turn.

5.1 Behavior near $w = -1 - \delta w$

In this subsection we study the pole in the neighborhood of $w = -1$. When $w \to -1 - \delta w$ with $0 < \delta w \ll 1$, we also expand $y \to 1 - \delta y$ (where $0 < \delta y \ll 1$) and find that

$$
\begin{aligned}
d_1 &\sim 4|m|^2\left((\mathrm{sgn}(\lambda)-1)\left(\delta w - 2\left(\frac{2}{\delta y}\right)^{\lambda}\right) + 2(\mathrm{sgn}(\lambda)+1)\right),\\
d_2 &\sim (\mathrm{sgn}(\lambda)+1)\left(2 - \left(\frac{2}{\delta y}\right)^{\lambda}\delta w\right) - 2\left(\frac{2}{\delta y}\right)^{\lambda}(\mathrm{sgn}(\lambda)-1)\,,\\
a_b &\sim 1-\delta y\,,\\
b_b &\sim -(2+\delta w)(\mathrm{sgn}(\lambda)+1-\delta y)^2\,,\\
c_b &\sim 4|m|^2(\mathrm{sgn}(\lambda)-1+\delta y)(\mathrm{sgn}(\lambda)(2+\delta w)+(2-\delta w)(1-\delta y))\,,\\
a_f &\sim 1-\delta y\,,\\
b_f &\sim 2\delta y(2+\delta w)(2-\delta y)\,,\\
c_f &\sim -4|m|^2(\mathrm{sgn}(\lambda)+1-\delta y)(\mathrm{sgn}(\lambda)(\delta w+2)+(2-\delta w)(1-\delta y))\,.
\end{aligned}
\tag{5.6}
$$

Let us first consider the case $\lambda > 0$. In this case d_1 equals $16m^2$ at leading order and so does not have a zero for δw and δy small. On the other hand

$$
d_2 \propto \left(2 - \left(\frac{2}{\delta y}\right)^{\lambda}\delta w\right)
$$

and so vanishes when

$$
\frac{\delta w}{2} = \left(\frac{\delta y}{2}\right)^{|\lambda|}, \quad \frac{\delta y}{2} = \left(\frac{\delta w}{2}\right)^{\frac{1}{|\lambda|}}. \tag{5.7}
$$

When $\lambda < 0$, d_2 is a monotonic function that never vanishes. However d_1 vanishes provided the condition (5.7) is met. It follows that the S matrix has a pole when (5.7) is satisfied for both signs of λ.

The pole in the S matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\to$ boson boson and the fermion fermion $\to$ fermion fermion S matrices, both these scattering processes have a pole at the value of y listed in (5.7). The residue of this pole is, however, significantly different in the four boson and four fermion scattering processes. Let us first consider the four boson scattering term. The residue of the pole is determined by b_b evaluated at (5.7) (in the case $\lambda > 0$) and c_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$
\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}}. \tag{5.8}
$$

In a similar manner the residue of the pole for four fermion scattering is determined by b_f evaluated at (5.7) (in the case $\lambda > 0$) and c_f evaluated at the same pole (in the case

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$\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1+|\lambda|}}{\delta w - 2\left(\frac{\delta y}{2}\right)^{|\lambda|}}\,. \tag{5.9}$$

Notice that while the residue of the pole for four fermion scattering is suppressed compared to the residue of the same pole for four boson scattering by a factor of $(\delta w)^{\frac{1}{|\lambda|}}$.

5.2 Pole near $y = 0$

There exists a critical value, $w = w_c(\lambda)$, at which both d_1 and d_2 have zeroes at $y = 0$. In order to locate w_c we expand d_1 and d_2 about $y = 0$. To linear order we find

$$d_1 = d_2 \sim y(\lambda \mathrm{sgn}(\lambda)(w-1) + 2)\,. \tag{5.10}$$

Clearly d_1 and d_2 have a common zero at $y = 0$ provided

$$w = w_c(\lambda) = 1 - \frac{2}{|\lambda|}\,. \tag{5.11}$$

In order to study this pole in the neighborhood of $w = w_c$ we set $w = w_c + \delta w$ (with $|\delta w| < 1$) near $y = \delta y$ (with $0 < \delta y \ll 1$); expanding in δw and δy we find

$$\begin{aligned} d_1 &\sim \frac{8|m|^2 \delta y\,(\delta w \lambda + 2\delta y(1-|\lambda|))}{\mathrm{sgn}(\lambda)}\,, \\ d_2 &\sim \frac{\delta y\,(\delta w \lambda - 2\delta y(1-|\lambda|))}{\mathrm{sgn}(\lambda)}\,, \\ n_b &\sim -\frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda}\,, \\ n_f &\sim \frac{512|m|^3 \sin(\pi\lambda)\delta y^2(-1+|\lambda|)}{\lambda}\,. \end{aligned} \tag{5.12}$$

The product $d_1 d_2$ vanishes when[44]

$$\delta y = \frac{|\lambda \delta w|}{2(1-|\lambda|)}, \quad \text{i.e.} \quad \delta y^2 = \frac{\lambda^2 \delta w^2}{4(1-|\lambda|)^2}\,. \tag{5.13}$$

The residue of the pole at $y = 0$ for any sign of λ is given by substituting (5.13) into the functions n_b and n_f in (5.12). We find the pole structure of the bosonic S matrix near $y = 0$ to be

$$\mathcal{T}_B \sim -\frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\,(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2)}\,. \tag{5.14}$$

In a similar manner we find the pole structure of the fermion S matrix near $y = 0$ to be

$$\mathcal{T}_F \sim \frac{64|m|\sin(\pi\lambda)(-1+|\lambda|)}{|\lambda|\,(\delta w^2\lambda^2 - 4\delta y^2(1-|\lambda|)^2)}\,. \tag{5.15}$$

[44] $d_1 d_2$ also vanishes quadratically at $\delta y = 0$. Note however that both n_b and n_f are proportional to δy^2. Consequently the factors of δy^2 cancel between the numerator and denominator.

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5.3 Behavior at $w \to -\infty$

We now turn to the analysis of the pole structure at $w \to -\infty$. This is easily achieved by setting $w = -\frac{1}{\delta w}$ with $0 < \delta w \ll 1$ and $y \to 1 - \delta y$ with $0 < \delta y \ll 1$. The various functions (5.2) in the S matrix (5.1) have the behavior

$$
\begin{aligned}
d_1 &\sim \frac{4|m|^2}{\delta w}\left((\delta y + \mathrm{sgn}(\lambda) - 1)\left(1 - \left(\frac{2}{\delta y}\right)^{\lambda}\right) + (\mathrm{sgn}(\lambda)+3)\delta w\right) , \\
d_2 &\sim \frac{1}{\delta w}\left((-\delta y + \mathrm{sgn}(\lambda) + 1) - \left(\frac{2}{\delta y}\right)^{\lambda}((\mathrm{sgn}(\lambda)-3)\delta w + \mathrm{sgn}(\lambda)+1)\right) , \\
a_b &\sim 1 - \delta y \, , \\
b_b &\sim -\frac{1}{\delta w}(\mathrm{sgn}(\lambda) + 1 - \delta y)^2 \, , \\
c_b &\sim -4|m|^2(\mathrm{sgn}(\lambda) - 1 + \delta y)\left(-\mathrm{sgn}(\lambda)\left(1 + \frac{1}{\delta w}\right) - \left(3 - \frac{1}{\delta w}\right)(1-\delta y)\right) , \\
a_f &\sim 1 - \delta y \, , \\
b_f &\sim \frac{\delta y}{\delta w}(2 - \delta y) \, , \\
c_f &\sim 4|m|^2(\mathrm{sgn}(\lambda) + 1 - \delta y)\left(-\mathrm{sgn}(\lambda)\left(1 + \frac{1}{\delta w}\right) - \left(3 - \frac{1}{\delta w}\right)(1-\delta y)\right) .
\end{aligned}
\tag{5.16}
$$

Let us first consider the case $\lambda > 0$. In this case d_2 is a monotonic function that never vanishes and so does not have a zero for δw and δy small. On the other hand

$$
d_1 \propto \left(\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}\right)
$$

and so vanishes when

$$
\delta w = \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|} , \qquad \delta y = \left(\frac{4\delta w}{2^{|\lambda|}}\right)^{\frac{1}{1-|\lambda|}} . \tag{5.17}
$$

When $\lambda < 0$, d_1 is a constant $-8m^2$. However d_2 vanishes provided the condition (5.17) is met. It follows that the S matrix has a pole when (5.17) is satisfied for both signs of λ.

The pole in the S matrix occurs due to the vanishing of the denominator $d_1 d_2$. As this denominator is the same for both the boson boson $\to$ boson boson and the fermion fermion $\to$ fermion fermion S matrices, both these scattering processes have a pole at the value of y listed in (5.17). The residue of this pole is different in the four boson and four fermion scattering processes as before. Let us first consider the four boson scattering term. The residue of the pole is determined by c_b evaluated at (5.17) (in the case $\lambda > 0$) and b_b evaluated at the same pole (in the case $\lambda < 0$). In either case we find the structure of the pole for four boson scattering to be

$$
\mathcal{T}_B \sim \frac{\left(\frac{\delta y}{2}\right)^{2-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}} . \tag{5.18}
$$

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In a similar manner the residue of the pole for four fermion scattering is determined by c_f evaluated at (5.7) (in the case $\lambda > 0$) and b_f evaluated at the same pole (in the case $\lambda < 0$). In either case we find that

$$\mathcal{T}_F \sim \frac{\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}{\delta w - \frac{1}{2}\left(\frac{\delta y}{2}\right)^{1-|\lambda|}}\,. \tag{5.19}$$

Notice that the residue of the pole for four boson scattering is supressed by a factor of $(\delta w)^{\frac{1}{1-|\lambda|}}$ compared to the residue for four fermion scattering.

5.4 Duality invariance

It is most interesting to note that the statements and results obtained in the above sections ((5.7), (5.11) and (5.17)) are all duality invariant. This is most transparent from the observation that under the duality transformation (2.12)[45]

$$\begin{aligned} d_1 &\leftrightarrow d_1\,, \\ d_2 &\leftrightarrow d_2\,. \end{aligned} \tag{5.20}$$

Hence the zeroes of d_1 and d_2 ((5.7) and (5.17)) should map to themselves, and w_c (5.11) should be duality invariant. Also recollect that under duality the bosonic and fermionic S matrices map to one another. Thus it is natural to expect that the pole in the bosonic S matrix at $w = -1$ (5.7) should map to the pole of the fermionic S matrix at $w = -\infty$ (5.17) and vice versa. Since both the bosonic and fermionic S matrices have a pole at $w = w_c$ (5.11) at $y = 0$, this pole should be self dual.

Upon using (2.12) on (5.11) it is straightforward to see that it is duality invariant. The slightly non-trivial part is the mapping of the two scaling regimes (5.7) and (5.17). It is straightforward to obtain the identification from $w = -\infty$ to $w = -1$ from (2.12)

$$-\frac{1}{\delta w_\infty} = \frac{3-(-1-\delta w_{-1})}{1+(-1-\delta w_{-1})} \sim -\frac{4}{\delta w_{-1}} \tag{5.21}$$

Using the above result in (5.17) and applying (2.12) for λ it is easy to check that (5.7) follows (and vice versa).

5.5 Scaling limit of the S matrix

In this subsection we discuss a particularly interesting near-threshold limit of the S-matrix. It was shown in [39] that in this limit the S matrices for the boson-boson and fermion-fermion reduce to the ones that are obtained by solving the Schrodinger equation with Amelino-Camelia-Bak boundary conditions [47, 48]. In this subsection we illustrate that the analysis of [39] applies for our results as well. We consider the near threshold region

$$y = 1 + \frac{k^2}{2m^2} \tag{5.22}$$

[45] Under duality transformation d_1 and d_2 transform into one another upto an overall non-zero factor. This overall factor is cancelled by an identical contribution from the duality transform of the numerator.

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with $k \ll 1$ and

$$w = -1 - \delta w \tag{5.23}$$

where $0 < \delta w \ll 1$. In the limit

$$k \to 0, \quad \delta w \to 0, \quad , \frac{k^2}{4m^2} \left(\frac{\delta w}{2}\right)^{-\frac{1}{|\lambda|}} = \text{fixed} \tag{5.24}$$

the J function in the bosonic S matrix ((5.1)) reduces to[46]

$$J_B = 8|m \sin(\pi\lambda)| \frac{1 + e^{i\pi|\lambda|} \frac{A_R}{k^{2|\lambda|}}}{1 - e^{i\pi|\lambda|} \frac{A_R}{k^{2|\lambda|}}}. \tag{5.25}$$

where

$$A_R = \frac{4^{|\lambda|}}{2} |m|^{2|\lambda|} \delta w \,. \tag{5.26}$$

Comparing our Lagrangian (2.11) with that of eq. 1.1 of [39] we make the parameter identifications

$$\delta w = \frac{\delta b_4}{8|m|\pi\lambda} \,.$$

Substituting δw in (5.26) we see that (5.25) matches exactly with eq. 1.12 of [39].

5.6 Effective theory near $w = w_c$?

As we have explained above, our theory develops a massless bound state at $w = w_c$; the mass of this bound state scales like $w - w_c$ in units of the mass of the scattering particles.[47] When $w - w_c \ll 1$ there is a separation of scales between the new bound state and all other excitations in our theory. In this regime the effective dynamics of the nearly massless particles should be governed by an autonomous quantum field theory that makes no reference to UV degrees of freedom. It seems likely that the superfield that creates the bound states is a real $\mathcal{N} = 1$ superfield. The fixed point that governs the dynamics of this field presumably has a single relevant deformation; as it was possible to approach this theory with a single fine tuning (setting $w = w_c$). These considerations suggest that the dynamics of the light bound state is governed by an $\mathcal{N} = 1$ Wilson-Fisher theory built out of a single real superfield. If this suggestion is correct it would imply that the long distance dynamics of the light bound states is independent of λ. Given that the bound states are gauge neutral this possibility does not seem absurd to us. It would be interesting to study this suggestion in future work.

6 Discussion

In this paper we have presented computations and conjectures for the all orders S matrix in the most general renormalizable $\mathcal{N} = 1$ Chern-Simons matter theory with a single

[46] Here we work in the regime $\sqrt{s} > 2m$ i.e $y > 1$ and hence the appearance of the factors of $e^{i\pi\lambda}$.

[47] We expect all of these results to continue to hold at finite N at least when N is large; in the rest of the discussion we assume that N is finite, and so the interactions between two bound state particles is not parametrically suppressed.

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fundamental matter multiplet. Our results are consistent with unitarity if we assume that the usual results of crossing symmetry are modified in precisely the manner proposed in [36], whereas the usual crossing symmetry rules are inconsistent with unitarity. We view this fact as a nontrivial consistency check of the crossing symmetry rules proposed in [36].

The 'particle-antiparticle' S matrix in the singlet channel conjectured in this paper has an interesting analytic structure. In a certain range of superpotential parameters the S matrix has a bound state pole; a one parameter tuning of superpotential parameters can be used to set the pole mass to zero. We find the existence of a massless bound state in a theory whose elementary excitations are all massive fascinating. It would be interesting to further investigate the low energy dynamics of these massless bound states. It would also be interesting to investigate if these bound states are 'visible' in the explicit results for the partition functions of Chern-Simons matter theories.

As we have explained in the previous section, our singlet sector particle-antiparticle S matrix has a simple non-relativistic limit. It would be useful to reproduce this scattering amplitude from the solution of a manifestly supersymmetric Schrodinger equation.

The results of this paper suggest many natural extensions and questions. First it would be useful to generalize the computations of this paper to the mass deformed $\mathcal{N} = 3$ and especially to the mass deformed $\mathcal{N} = 6$ susy gauge theories (the later is necessarily a $\mathrm{U}(N) \times \mathrm{U}(M)$ theory; the methods of this paper are likely to be useful in the limit $N \to \infty$ with M held fixed). This generalization should allow us to make contact with earlier studies of scattering in ABJ theory [14–20] that were performed arbitrary values of M and N but perturbatively (to given loop order) in λ.

At the $\mathcal{N} = 2$ point the S matrices presented in this paper are tree level exact in the three non anyonic channels, and depend on λ in a very simple way in the singlet channel. It is possible that this very simple result can be deduced in a more structural manner using only general principles and $\mathcal{N} = 2$ supersymmetry. It would be interesting if this were the case.

As an intermediate step in the computation of the S matrix we evaluated the off shell four point function of four superfields. This four point correlator was rather complicated in the general $\mathcal{N} = 1$ theory, but extremely simple at the $\mathcal{N} = 2$ point. The four point correlator (or sum of ladder diagrams) is a useful intermediate piece in the evaluation of two, three and four point functions of gauge invariant operators [26, 29, 40, 42]. The simplicity of the $\mathcal{N} = 2$ results suggest that it would be rather easy to explicitly evaluate such correlators, at least in special kinematic limits. Such computations could be used as independent checks of duality as well as well as inputs into $\mathcal{N} = 2$ generalizations of the Maldacena-Zhiboedov solutions of Chern-Simons fundamental matter theories [23, 24].

All of the computations in this paper have been performed under the assumption $\lambda m \geq 0$. Atleast naively all of the checks of duality (including earlier checks involving the partion function) fail when $\lambda m < 0$. It would be interesting to understand why this is the case. It is possible that our theory undergoes a phase transition as λm changes sign (see [30, 33] for related discussion). It would be interesting to understand this better.

We believe that the results of this paper put the crossing symmetry relations conjectured in [36] on a firm footing. It would be interesting to find a rigorous proof of these

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crossing relations, and even more interesting to hit upon a plausible generalization of these relations to finite N and k. From a traditional perturbative point of view the modified crossing symmetry rules are presumably related to infrared divergences. It thus seems possible that one route to a proof and generalization of these relations lies in a detailed study of the infrared divergences of the relevant Feynman graphs. We hope to return to several of these questions in the future.

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A Notations and conventions

A.1 Gamma matrices

In this section, we list the various notations and conventions used in this paper. We follow those of [49]. We list them here for convenience.

The metric signature is $\eta_{\mu\nu} = \{-,+,+\}$. In three dimensions the Lorentz group is $SL(2,\mathbb{R})$ and it acts on two component real spinors ψ^α, where α are the spinor indices. A vector is represented by either a real and symmetric spinor $V_{\alpha\beta}$ or a symmetric traceless spinor V_α^β, where $V_{\alpha\beta} = V_\mu \gamma^\mu_{\alpha\beta}$. We will choose our gamma matrices in the real and symmetric form [50]

$$\gamma^\mu_{\alpha\beta} = \{\mathbb{I}, \sigma^3, \sigma^1\}\,. \tag{A.1}$$

The charge conjugation matrix $C_{\alpha\beta}$ is used to raise and lower the spinor indices

$$C_{\alpha\beta} = -C_{\beta\alpha} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -C^{\alpha\beta}\,. \tag{A.2}$$

In the above, note that $C_{\beta\alpha} = C^T$ and $C^{\alpha\beta} = (C^T)^{-1}$. It follows that

$$C_{\alpha\gamma}C^{\gamma\beta} = -\delta_\alpha^{\ \beta}\,, \tag{A.3}$$

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where $\delta_\alpha^{\ \beta}$ is the usual identity matrix. The spinor indices are raised and lowered using the NW-SE convention

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta\,, \psi_\alpha = \psi^\beta C_{\beta\alpha}\,. \tag{A.4}$$

We also use the notation $\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha = i\psi^+\psi^-$. Note that ψ^2 is Hermitian. Since ψ^α is real, it is clear that ψ_α is imaginary since the charge conjugate matrix is imaginary.

The Clifford algebra is defined using the matrices $(\gamma^\mu)_\alpha^{\ \beta}$ and these can be obtained by raising the indices using $C^{\alpha\beta}$ as illustrated above

$$(\gamma^\mu)_\alpha^{\ \beta} = \{\sigma^2, -i\sigma^1, i\sigma^3\}\,. \tag{A.5}$$

Note that these matrices are purely imaginary. Choosing the $\gamma^\mu_{\alpha\beta}$ as real and symmetric always yields this and vice versa. Our $\mu = 0, 1, 3$, since at some point we will do an euclidean rotation from the $\mu = 0$ direction to $\mu = 2$. It is clear that $(\gamma^0)^2 = 1, (\gamma^1)^2 = -1, (\gamma^3)^2 = -1$, therefore with our metric conventions the Clifford algebra is satisfied by

$$(\gamma^\mu)_\alpha^{\ \tau}(\gamma^\nu)_\tau^{\ \beta} + (\gamma^\nu)_\alpha^{\ \tau}(\gamma^\mu)_\tau^{\ \beta} = -2\eta^{\mu\nu}\delta_\alpha^{\ \beta}\,. \tag{A.6}$$

Another very useful relation is

$$[\gamma^\mu, \gamma^\nu] = -2i\epsilon^{\mu\nu\rho}\gamma_\rho\,, \epsilon^{013} = -1\,. \tag{A.7}$$

For completion we also note that

$$(\gamma^\mu)^{\alpha\beta} = \{\mathbb{I}, \sigma^3, \sigma^1\}\,. \tag{A.8}$$

As a consequence of the Clifford algebra (A.6), we get a minus sign in the trace

$$k_\alpha^{\ \beta}k_\beta^{\ \alpha} = -2k^2\,. \tag{A.9}$$

The Euclidean counterpart of (A.6) is obtained by the standard Euclidean rotation $\gamma^0 \to i\gamma^2$

$$(\gamma^\mu)_\alpha^{\ \beta} = \{i\sigma^2, -i\sigma^1, i\sigma^3\}\,,\ \mu = 2, 1, 3, \tag{A.10}$$

and they satisfy the Euclidean Clifford algebra

$$(\gamma^\mu)_\alpha^{\ \tau}(\gamma^\nu)_\tau^{\ \beta} + (\gamma^\nu)_\alpha^{\ \tau}(\gamma^\mu)_\tau^{\ \beta} = -2\delta^{\mu\nu}\delta_\alpha^{\ \beta}\,. \tag{A.11}$$

where $\delta_{\mu\nu} = (+, +, +)$.

A.2 Superspace

The two component Grassmann parameters θ that appear in various places in superspace have the properties

$$\begin{gathered}\int d\theta = 0\,, \qquad \int d\theta\theta = 1\,, \quad \int d^2\theta\theta^2 = -1\,, \qquad \int d^2\theta\theta^\alpha\theta^\beta = C^{\alpha\beta}\,, \\ \frac{\partial\theta^\alpha}{\partial\theta^\beta} = \delta_\beta^{\ \alpha}\,, \quad C^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}\frac{\partial}{\partial\theta^\alpha}\theta^2 = -2\,, \quad \theta^\alpha\theta^\beta = -C^{\alpha\beta}\theta^2\,, \qquad \theta_\alpha\theta_\beta = -C_{\alpha\beta}\theta^2\,. \end{gathered} \tag{A.12}$$

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The definition of the delta function in superspace follows from the relation

$$\int d^2\theta\theta^2 = -1 \implies \delta^2(\theta) = -\theta^2\,. \tag{A.13}$$

Formally we write

$$\delta^2(\theta_1-\theta_2) = -(\theta_1-\theta_2)^2 = -(\theta_1^2+\theta_2^2-\theta_1\theta_2). \tag{A.14}$$

The superspace derivatives are defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\partial_{\alpha\beta}\,, D^\alpha = C^{\alpha\beta}D_\beta\,. \tag{A.15}$$

We will mostly use the momentum space version of the above in which we replace $i\partial_{\alpha\beta}\to k_{\alpha\beta}$

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \theta^\beta k_{\alpha\beta}\,. \tag{A.16}$$

Note that the choice of the real and symmetric basis in (A.1) makes the momentum operator Hermitian. The superspace derivatives satisfy the algebra

$$\{D_\alpha, D_\beta\} = 2k_{\alpha\beta}\,. \tag{A.17}$$

The tracelessness of $(\gamma^\mu)_\alpha^{\ \beta}$ implies that

$$\{D^\alpha, D_\alpha\} = 0\,. \tag{A.18}$$

Care has to be taken when integrating by parts with superderivatives due to their anti-commuting nature. From the expression for D_α we can construct

$$D^2 = \frac{1}{2}D^\alpha D_\alpha = \frac{1}{2}\left(C^{\beta\alpha}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + 2\theta^\alpha k_\alpha^{\ \beta}\frac{\partial}{\partial\theta^\beta} + 2\theta^2k^2\right). \tag{A.19}$$

From the above it is easy to verify

$$\begin{aligned}(D^2)^2 &= -k^2,\\ D^2D_\alpha &= -D_\alpha D^2 = k_{\alpha\beta}D^\beta\\ D^\alpha D_\beta D_\alpha &= 0\,.\end{aligned} \tag{A.20}$$

using the properties given in (A.12). Yet another extremely useful relation is the action of the superderivative square (A.19) on the delta function (A.14)

$$D^2_{\theta_1,k}\delta^2(\theta_1-\theta_2) = 1-\theta_1^\alpha\theta_2^\beta k_{\alpha\beta} - \theta_1^2\theta_2^2k^2 = \exp(-\theta_1^\alpha\theta_2^\beta k_{\alpha\beta})\,. \tag{A.21}$$

We will often suppress the spinor indices in the exponential with the understanding that the spinor indices are contracted as indicated above. Some useful formulae are

$$\begin{aligned}\delta^2(\theta_1-\theta_2)\delta^2(\theta_2-\theta_1) &= 0\,,\\ \delta^2(\theta_1-\theta_2)D^\alpha_{\theta_2,k}\delta^2(\theta_2-\theta_1) &= 0\,,\\ \delta^2(\theta_1-\theta_2)D^2_{\theta_2,k}\delta^2(\theta_2-\theta_1) &= \delta^2(\theta_1-\theta_2)\,,\end{aligned} \tag{A.22}$$

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and the transfer rule

$$D_\alpha^{\theta_1,p}\delta^2(\theta_1-\theta_2) = -D_\alpha^{\theta_2,-p}\delta^2(\theta_2-\theta_1)\,. \tag{A.23}$$

The supersymmetry generators

$$Q_\alpha^{\theta,k} = i\left(\frac{\partial}{\partial\theta^\alpha} - \theta^\beta k_{\alpha\beta}\right)\,, \tag{A.24}$$

satisfy the anticommutation relations

$$\begin{aligned}\{Q_\alpha, Q_\beta\} &= 2k_{\alpha\beta}\,,\\ \{Q_\alpha, D_\beta\} &= 0\,.\end{aligned} \tag{A.25}$$

It is also clear that the transfer rule (A.23) is the statement that the delta function of θ is a supersymmetric invariant.

A.3 Superfields

The scalar superfield $\Phi(x,\theta)$ contains a complex scalar ϕ, a complex fermion ψ^α, and a complex auxiliary field F. The vector superfield $\Gamma^\alpha(x,\theta)$ consists of the gauge field $V_{\alpha\beta}$, the gaugino λ^α, an auxiliary scalar B and an auxiliary fermion χ^α. The following superfield expansions are used repeatedly in several places. We list them here for easy reference.

$$\begin{aligned}\Phi &= \phi + \theta\psi - \theta^2 F\,,\\ \bar\Phi &= \bar\phi + \theta\bar\psi - \theta^2\bar F\,,\\ \bar\Phi\Phi &= \bar\phi\phi + \theta^\alpha(\bar\phi\psi_\alpha + \bar\psi_\alpha\phi) - \theta^2(\bar F\phi + \bar\phi F + \bar\psi\psi)\,,\\ D_\alpha\Phi &= \psi_\alpha - \theta_\alpha F + i\theta^2\partial_\alpha{}^\beta\psi_\beta + i\theta^\beta\partial_{\alpha\beta}\phi\,,\\ D^\alpha\bar\Phi D_\alpha\Phi\big|_{\theta^2} &= \theta^2(2\bar F F + 2i\bar\psi^\alpha\partial_\alpha{}^\beta\psi_\beta - 2\partial\bar\phi\partial\phi)\,,\\ D^2_{q,\theta}(\bar\Phi\Phi) &= (\bar\phi F + \bar F\phi + \bar\psi\psi) + \theta^\alpha q_\alpha{}^\beta(\bar\phi\psi + \bar\psi\phi) + \theta^2 q^2(\bar\phi\phi)^2\,,\\ \Gamma^\alpha &= \chi^\alpha - \theta^\alpha B + i\theta^\beta A_\beta^\alpha - \theta^2(2\lambda^\alpha - i\partial^{\alpha\beta}\chi_\beta)\,.\end{aligned} \tag{A.26}$$

B A check on the constraints of supersymmetry on S matrices

In section 2.4 we demonstrated that the manifestly supersymmetric scattering of any $\mathcal{N}=1$ theory in three dimensions is described by two independent functions. In this section, we directly verify this result in theories whose offshell effective action takes the form (3.27) with the function V that takes the particular supersymmetric form (3.31) (and so is determined by four unspecified functions A, B C and D).

We wish to use (3.27) to study scattering. In order to do this we evaluate (3.27) with the fields Φ and $\bar\Phi$ in that action chosen to be the most general linearized onshell solutions to the equations of motion. In this appendix we focus on a particular scattering process — scattering in the adjoint channel. At leading order in the large N limit we can focus on this channel by choosing the solution for Φ_m and $\bar\Phi^m$ in (3.27) to be positive energy solutions (representing initial states), while $\bar\Phi^m$ and Φ_n are expanded in negative energy solutions (representing final states). The negative and positive energy solutions

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are both allowed to be an arbitrary linear combination of bosonic and fermionic solutions. Plugging these solutions into (3.27) yields a functional of the coefficients of the bosonic and fermionic solutions in the four superfields in (3.27). The coefficients of various terms in this functional are simply the S matrices. For instance the coefficient of the term proportional to the product of four bosonic modes is the four boson scattering amplitude, etc.

Let us schematically represent the scattering process we study by

$$\begin{pmatrix} \Phi(\theta_1, p_1) \\ \bar{\Phi}(\theta_2, p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\Phi}(\theta_3, p_3) \\ \Phi(\theta_4, p_4) \end{pmatrix} \tag{B.1}$$

where the l.h.s. represents the in-states and the r.h.s. represents the out-states. The momentum assignments in (3.27) are

$$p_1 = p + q\,,\ \ p_2 = -k - q\,,\ \ p_3 = p\,,\ \ p_4 = -k\,. \tag{B.2}$$

In component form (B.1) encodes the following S matrices

$$\begin{aligned}
&\mathcal{S}_B : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix}, \quad \mathcal{S}_F : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix} \\
&H_1 : \begin{pmatrix} \phi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \psi(p_4) \end{pmatrix}, \quad H_2 : \begin{pmatrix} \psi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \phi(p_4) \end{pmatrix} \\
&H_3 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix}, \quad H_4 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix} \\
&H_5 : \begin{pmatrix} \phi(p_1) \\ \bar{\psi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\psi}(p_3) \\ \phi(p_4) \end{pmatrix}, \quad H_6 : \begin{pmatrix} \psi(p_1) \\ \bar{\phi}(p_2) \end{pmatrix} \to \begin{pmatrix} \bar{\phi}(p_3) \\ \psi(p_4) \end{pmatrix}.
\end{aligned} \tag{B.3}$$

These S matrix elements are all obtained by the process spelt out above in terms of the four unknown functions A, B, C, D (which we will take to be arbitrary and unrelated). The functions A,B,C,D are to be evaluated at the onshell conditions that follow from taking the momenta onshell, but that will play no role in what follows.

It is not difficult to demonstrate that the boson-boson $\to$ boson boson and the fermion-fermion $\to$ fermion fermion S matrices are given in terms of the functions A, B, C and D by[48]

$$\begin{aligned}
\mathcal{S}_B &= (-4iAm + 4Bm^2 - Bq_3^2 - q_3(Ck_- + Dp_-))\,, \\
\mathcal{S}_F &= (BC^{\beta\alpha}C^{\delta\gamma} - iC\ C^{\beta\alpha}C^{+\gamma}C^{+\delta} + iDC^{\delta\gamma}C^{+\alpha}C^{+\beta})\bar{u}_\alpha(p_3)u_\beta(p_1)v_\gamma(p_2)\bar{v}_\delta(p_4) \\
&= -B(4m^2 + q_3^2) + Ck_-(2im - q_3) - Dp_-(q_3 + 2im)\,.
\end{aligned} \tag{B.5}$$

[48]For the T channel we have used

$$\begin{aligned}
v_\alpha(-k) &= \begin{pmatrix} -\sqrt{k_+} \\ \frac{(q_3 - 2im)}{2\sqrt{k_+}} \end{pmatrix}, \quad & \bar{v}^\alpha(-k-q) &= \begin{pmatrix} -\frac{2m + iq_3}{2\sqrt{k_+}} & i\sqrt{k_+} \end{pmatrix} \\
u_\alpha(p+q) &= \begin{pmatrix} -i\sqrt{p_+} \\ \frac{2m - iq_3}{2\sqrt{p_+}} \end{pmatrix}, & \bar{u}^\alpha(p) &= \begin{pmatrix} -\frac{(2im + q_3)}{2\sqrt{p_+}} & -\sqrt{p_+} \end{pmatrix}.
\end{aligned} \tag{B.4}$$

The S matrices for the remaining processes in (B.3) are also easily obtained: we find

$$H_i = a_i \mathcal{S}_B + b_i \mathcal{S}_F \tag{B.6}$$

where the coefficients are given by

$$\begin{aligned}
a_1 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(p-k)_-+2im(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}, & b_1 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(k-p)_-+2im(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_2 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(p-k)_-+2im(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}, & b_2 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(k-p)_-+2im(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_3 &= -\frac{2m+iq_3}{4m}, & b_3 &= \frac{2m+iq_3}{4m}\\
a_4 &= \frac{2m-iq_3}{4m}, & b_4 &= -\frac{2m-iq_3}{4m}\\
a_5 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(k+p)_--2im(k-p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}, & b_5 &= -\frac{i\left(4m^2+q_3^2\right)\left(2m(k-p)_--iq_3(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}\\
a_6 &= \frac{i\left(4m^2+q_3^2\right)\left(2m(k-p)_-+iq_3(k+p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}, & b_6 &= \frac{\left(4m^2+q_3^2\right)\left(q_3(k+p)_-+2im(k-p)_-\right)}{32mk_-p_-\sqrt{k_+p_+}}
\end{aligned} \tag{B.7}$$

The above set of coefficients match with the coefficients directly evaluated from (2.38) and (2.39). This is a consistency check of the results of section 2.4.

For the $\mathcal{N}=2$ theory the S matrix (2.37) should also obey an additional constraint (see section C) that relates $\mathcal{S}_B$ and $\mathcal{S}_F$ through (C.24). For the T channel this relation was evaluated in (3.92), substituting this in (B.6) it is easy to verify that the $\theta_2\theta_3$ and $\theta_1\theta_4$ terms in (2.37)

$$H_5 : \begin{pmatrix} \phi(p_1) \\ \bar\psi(p_2) \end{pmatrix} \to \begin{pmatrix} \bar\psi(p_3) \\ \phi(p_4) \end{pmatrix}, \quad H_6 : \begin{pmatrix} \psi(p_1) \\ \bar\phi(p_2) \end{pmatrix} \to \begin{pmatrix} \bar\phi(p_3) \\ \psi(p_4) \end{pmatrix} \tag{B.8}$$

vanish for the $\mathcal{N}=2$ theory. This is consistent with the fact that the corresponding terms in the tree level component Lagrangian (2.11) vanish at the $\mathcal{N}=2$ point $w=1$.

C Manifest $\mathcal{N}=2$ supersymmetry invariance

In this appendix we discuss the general constraints on the S matrix obtained by imposing $\mathcal{N}=2$ supersymmetry. In subsection 2.4 we have already solved the constraints coming from $\mathcal{N}=1$ supersymmetry. As an $\mathcal{N}=2$ theory is in particular also $\mathcal{N}=1$ supersymmetric, the results of this appendix will be a specialization of those of subsection 2.4.

In the case of $\mathcal{N}=2$, we have to recall the notion of chirality. A 'chiral' (antichiral) $\mathcal{N}=2$ superfield Φ is defined as

$$\bar D_\alpha \Phi = 0, \qquad D_\alpha \bar\Phi = 0. \tag{C.1}$$

We define the following:

$$\theta_\alpha = \frac{1}{\sqrt2}(\theta_\alpha^{(1)} - i\theta_\alpha^{(2)}), \qquad \bar\theta_\alpha = \frac{1}{\sqrt2}(\theta_\alpha^{(1)} + i\theta_\alpha^{(2)}). \tag{C.2}$$

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Where the superscripts (1) and (2) indicate the two (real) copies of the $\mathcal{N} = 1$ superspace. With these definitions, we can define the supercharges

$$Q_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha^{(1)} + iQ_\alpha^{(2)}) = i\left(\frac{\partial}{\partial\theta^\alpha} - i\bar{\theta}^\beta\partial_{\beta\alpha}\right), \tag{C.3}$$

$$\bar{Q}_\alpha = \frac{1}{\sqrt{2}}(Q_\alpha^{(1)} - iQ_\alpha^{(2)}) = i\left(\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta\partial_{\beta\alpha}\right). \tag{C.4}$$

Likewise, we can define the supercovariant derivative operators

$$D_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(1)} + iD_\alpha^{(2)}) = \left(\frac{\partial}{\partial\theta^\alpha} + i\bar{\theta}^\beta\partial_{\beta\alpha}\right), \tag{C.5}$$

$$\bar{D}_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^{(1)} - iD_\alpha^{(2)}) = \left(\frac{\partial}{\partial\bar{\theta}^\alpha} + i\theta^\beta\partial_{\beta\alpha}\right). \tag{C.6}$$

The solutions to the constraints (C.1) for (off-shell) chiral and anti-chiral fields are

$$\Phi = \phi + \sqrt{2}\theta\psi - \theta^2 F + i\theta\bar{\theta}\partial\phi - i\sqrt{2}\theta^2(\bar{\theta}\partial\!\!\!/\psi) + \theta^2\bar{\theta}^2\partial^2\phi, \tag{C.7}$$

$$\bar{\Phi} = \bar{\phi} + \sqrt{2}\bar{\theta}\bar{\psi} - \bar{\theta}^2\bar{F} - i\theta\bar{\theta}\partial\bar{\phi} - i\sqrt{2}\bar{\theta}^2(\theta\partial\!\!\!/\bar{\psi}) + \theta^2\bar{\theta}^2\partial^2\bar{\phi}. \tag{C.8}$$

Here $\theta\bar{\theta}\partial\phi = \theta^\alpha\bar{\theta}^\beta\partial_{\alpha\beta}$ and $\bar{\theta}\partial\!\!\!/\psi = \bar{\theta}^\alpha\partial_\alpha{}^\beta\psi_\beta$ and so on.

In the context of the current paper the chiral matter superfield transforms in the fundamental representation of the gauge group while the antichiral matter superfield transforms in the antifundamental representation of the gauge group. It follows that it is impossible to add a gauge invariant quadratic superpotential to our action (recall that an $\mathcal{N} = 2$ superpotential can only depend on chiral multiplets) in order to endow our fields with mass. However it is possible to make the matter fields massive while preserving $\mathcal{N} = 2$ supersymmetry; the fields can be made massive using a D term.

As our theory has no superpotential, it follows that $F = \bar{F} = 0$ on shell. We are interested in the action of supersymmetry on the on-shell component fields ϕ ($\bar{\phi}$) which are defined as

$$\phi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}}\left[a(\mathbf{p})e^{ip\cdot x} + a^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{C.9}$$

$$\bar{\phi}(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}}\left[a^c(\mathbf{p})e^{ip\cdot x} + a^\dagger(\mathbf{p})e^{-ip\cdot x}\right]. \tag{C.10}$$

Likewise, for ψ ($\psi^\dagger$) we have

$$\psi(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}}\left[u_\alpha(\mathbf{p})\alpha(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^{c\dagger}(\mathbf{p})e^{-ip\cdot x}\right], \tag{C.11}$$

$$\psi^\dagger(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2p^0}}\left[u_\alpha(\mathbf{p})\alpha^c(\mathbf{p})e^{ip\cdot x} + v_\alpha(\mathbf{p})\alpha^\dagger(\mathbf{p})e^{-ip\cdot x}\right]. \tag{C.12}$$

In order to obtain this action we used the transformation properties listed in equations F.16-F.20 of [27] and then specialized to the onshell limit.[49] The results may be summarized as

[49]Note that the action of Q_α on the chiral field Φ is different from the action on the anti-chiral field $\bar{\Phi}$. Similar remarks apply for $\bar{Q}_\alpha$.

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follows. As before, we define the (super) creation and annihilation operators

$$A(\mathbf{p}) = a(\mathbf{p}) + \alpha(\mathbf{p})\theta, \qquad A^c(\mathbf{p}) = a^c(\mathbf{p}) + \alpha^c(\mathbf{p})\theta, \tag{C.13}$$

$$A^\dagger(\mathbf{p}) = a^\dagger(\mathbf{p}) + \theta\alpha^\dagger(\mathbf{p}), \qquad A^{c\dagger}(\mathbf{p}) = a^{c\dagger}(\mathbf{p}) + \theta\alpha^{c\dagger}(\mathbf{p}). \tag{C.14}$$

The action of Q_α (and $\bar{Q}_\alpha$) on A and $A^\dagger$ is

$$\begin{aligned}
[Q_\alpha, A(\mathbf{p})] &= -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \qquad [\bar{Q}_\alpha, A(\mathbf{p})] = i\sqrt{2}u^*_\alpha(\mathbf{p})\theta, \\
[Q_\alpha, A^\dagger(\mathbf{p})] &= i\sqrt{2}v^*_\alpha(\mathbf{p})\theta, \qquad [\bar{Q}_\alpha, A^\dagger(\mathbf{p})] = i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}.
\end{aligned} \tag{C.15}$$

Similarly, the action of Q_α (and $\bar{Q}_\alpha$) on A^c and $A^{c\dagger}$ is

$$\begin{aligned}
[Q_\alpha, A^c(\mathbf{p})] &= i\sqrt{2}u^*_\alpha(\mathbf{p})\theta, \qquad [\bar{Q}_\alpha, A^c(\mathbf{p})] = -i\sqrt{2}u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \\
[Q_\alpha, A^{c\dagger}(\mathbf{p})] &= i\sqrt{2}v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}, \qquad [\bar{Q}_\alpha, A^{c\dagger}(\mathbf{p})] = i\sqrt{2}v^*_\alpha(\mathbf{p})\theta.
\end{aligned} \tag{C.16}$$

It is clear from (C.15) that $(Q_\alpha + \bar{Q}_\alpha)/\sqrt{2}$ produces the action of the first supercharge $Q^{(1)}_\alpha$, which we have seen earlier. That this action produces the correct differential operator given earlier is obvious as well. Therefore, in order to obtain the second supercharge $Q^{(2)}_\alpha$, we simply operate with the other linear combination $(Q_\alpha - \bar{Q}_\alpha)/i\sqrt{2}$.

Note that for the $\mathcal{N} = 1$ case, it doesn't matter if we used $A^\dagger$ or $A^{c\dagger}$ for the initial states (A or A^c for the final states), as is clear from (C.16). This agrees with the fact that the linear combination $(Q_\alpha + \bar{Q}_\alpha)/\sqrt{2}$ produces the same equation on all S matrix elements. However other linear combinations of the two $\mathcal{N} = 2$ supersymmetries act differently on A and A^c, and so the constraints of $\mathcal{N} = 2$ supersymmetry are different depending on which scattering processes we consider.

C.1 Particle-antiparticle scattering

Let us first study the invariance of the following S matrix element

$$S(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) = \langle 0|A_4(\mathbf{p}_4, \theta_4)A^c_3(\mathbf{p}_3, \theta_3)A^\dagger_2(\mathbf{p}_2, \theta_2)A^{c\dagger}_1(\mathbf{p}_1, \theta_1)|0\rangle. \tag{C.17}$$

In the context of our paper, this is the S matrix for particle-antiparticle scattering. The full $\mathcal{N} = 2$ invariance of the S matrix is expressed as

$$\left(\sum_{i=1}^4 Q^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0, \text{ and } \left(\sum_{i=1}^4 \bar{Q}^i_\alpha(\mathbf{p}_i, \theta_i)\right) S(\mathbf{p}_i, \theta_i) = 0. \tag{C.18}$$

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The above conditions (C.18) produce the following constraints for the S matrix element (C.17)

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0 \Rightarrow \tag{C.19}$$
$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv^*(\mathbf{p}_2)\theta_2+iu^*_\alpha(\mathbf{p}_3)\theta_3-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right)S(\mathbf{p}_i,\theta_i)=0,$$
$$\left(\sum_{i=1}^{4} \bar{Q}^i_\alpha(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0 \Rightarrow$$
$$\left(iv^*_\alpha(\mathbf{p}_1)\theta_1+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}+iu^*_\alpha(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0.$$

We check in what follows that the combination

$$\left(\frac{1}{\sqrt{2}}\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i,\theta_i)+\bar{Q}^i_\alpha(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0 \tag{C.20}$$

produces the same equation (and therefore solution) of $\mathcal{N}=1$ which we have already found. We easily find that this gives

$$\begin{aligned}\Bigg(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}&+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\\ &+iv^*_\alpha(\mathbf{p}_1)\theta_1+iv^*_\alpha(\mathbf{p}_2)\theta_2+iu^*_\alpha(\mathbf{p}_3)\theta_3+iu^*_\alpha(\mathbf{p}_4)\theta_4\Bigg)S(\mathbf{p}_i,\theta_i)=0.\end{aligned} \tag{C.21}$$

Now, we turn to the other linear combination, which is

$$\left(\frac{1}{i\sqrt{2}}\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i,\theta_i)-\bar{Q}^i_\alpha(\mathbf{p}_i,\theta_i)\right) S(\mathbf{p}_i,\theta_i)=0. \tag{C.22}$$

This readily gives the differential equation

$$\begin{aligned}\Bigg(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}&-iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}+iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\\ &-iv^*_\alpha(\mathbf{p}_1)\theta_1+iv^*_\alpha(\mathbf{p}_2)\theta_2+iu^*_\alpha(\mathbf{p}_3)\theta_3-iu^*_\alpha(\mathbf{p}_4)\theta_4\Bigg)S(\mathbf{p}_i,\theta_i)=0.\end{aligned} \tag{C.23}$$

The equation (C.21) is the same as it was for the $\mathcal{N}=1$ theory, whereas the second equation (C.23) must be obeyed by the same S matrix in the $\mathcal{N}=2$ point. Thus (C.28) is an additional constraint obeyed by the $\mathcal{N}=2$ S matrix (2.37). It follows that (C.23) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{12}v_\alpha(\mathbf{p}_1)-C_{23}u_\alpha(\mathbf{p}_3)+C_{24}u_\alpha(\mathbf{p}_4)+v^*_\alpha(\mathbf{p}_2)\right)=\mathcal{S}_F(C^*_{13}u_\alpha(\mathbf{p}_4)+C^*_{14}u_\alpha(\mathbf{p}_3)+C^*_{34}v_\alpha(\mathbf{p}_1)) \tag{C.24}$$

Thus, the $\mathcal{N}=2$ S matrix for particle-antiparticle scattering consists of only one independent function, with the other related by (C.24).

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C.2 Particle-particle scattering

Now, consider the other S matrix element (which was considered in the previous $\mathcal{N}=1$ computation)

$$S(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=\langle 0|A_4(\mathbf{p}_4,\theta_4)A_3(\mathbf{p}_3,\theta_3)A_2^\dagger(\mathbf{p}_2,\theta_2)A_1^\dagger(\mathbf{p}_1,\theta_1)|0\rangle. \quad \text{(C.25)}$$

The conditions (C.18) produce the following for the S matrix element (C.25)

$$\left(\sum_{i=1}^{4} Q_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow \quad \text{(C.26)}$$
$$\left(iv_\alpha^*(\mathbf{p}_1)\theta_1+iv^*(\mathbf{p}_2)\theta_2-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right)S(\mathbf{p}_i,\theta_i)=0,$$
$$\left(\sum_{i=1}^{4} \bar{Q}_\alpha^i(\mathbf{p}_i,\theta_i)\right)S(\mathbf{p}_i,\theta_i)=0\Rightarrow$$
$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}+iu_\alpha^*(\mathbf{p}_3)\theta_3+iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0.$$

For the combination (C.20) we get

$$\left(iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}+iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right.$$
$$\left.+iv_\alpha^*(\mathbf{p}_1)\theta_1+iv_\alpha^*(\mathbf{p}_2)\theta_2+iu_\alpha^*(\mathbf{p}_3)\theta_3+iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0, \quad \text{(C.27)}$$

and for the combination (C.22) we have

$$\left(-iv_\alpha(\mathbf{p}_1)\frac{\overrightarrow{\partial}}{\partial\theta_1}-iv_\alpha(\mathbf{p}_2)\frac{\overrightarrow{\partial}}{\partial\theta_2}-iu_\alpha(\mathbf{p}_3)\frac{\overrightarrow{\partial}}{\partial\theta_3}-iu_\alpha(\mathbf{p}_4)\frac{\overrightarrow{\partial}}{\partial\theta_4}\right.$$
$$\left.+iv_\alpha^*(\mathbf{p}_1)\theta_1+iv_\alpha^*(\mathbf{p}_2)\theta_2-iu_\alpha^*(\mathbf{p}_3)\theta_3-iu_\alpha^*(\mathbf{p}_4)\theta_4\right)S(\mathbf{p}_i,\theta_i)=0. \quad \text{(C.28)}$$

Similar to the particle-antiparticle case discussed in the previous section. The equation (C.27) is the same as it was for the $\mathcal{N}=1$ theory, whereas the second equation (C.28) must be obeyed by the same S matrix in the $\mathcal{N}=2$ point. It follows that (C.28) gives a relation between $\mathcal{S}_B$ and $\mathcal{S}_F$

$$\mathcal{S}_B\left(C_{13}u_\alpha(\mathbf{p}_3)+C_{14}u_\alpha(\mathbf{p}_4)+C_{12}v_\alpha(\mathbf{p}_2)+v_\alpha^*(\mathbf{p}_1)\right)=\mathcal{S}_F(C_{24}^*u_\alpha(\mathbf{p}_3)-C_{23}^*u_\alpha(\mathbf{p}_4)+C_{34}^*v_\alpha(\mathbf{p}_2)) \quad \text{(C.29)}$$

The $\mathcal{N}=2$ S matrix for particle-particle scattering consists of only one independent function, with the other related by (C.29).

Thus in the $\mathcal{N}=2$ theory the S matrix is only made of one independent function. Note that the results of this section are true for *any* three dimensional $\mathcal{N}=2$ theory. It simply follows from the supersymmetric ward identity (C.18) and is independent of the details of the theory.

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D Identities for S matrices in onshell superspace

In this subsection we demonstrate that the product of two supersymmetric S matrices is supersymmetric. In other words we demonstrate that

$$\left(\sum_{i=1}^{4} Q^i_\alpha(\mathbf{p}_i,\theta_i)\right) S_1 \star S_2 = 0. \tag{D.1}$$

provided S_1 and S_2 independently obey the same equation.

This can be analyzed as follows. We have the invariance (differential) equation for S_1 and S_2

$$\begin{aligned}\left(\overrightarrow{Q}_{\tilde{v}(\mathbf{p}_1)}+\overrightarrow{Q}_{\tilde{v}(\mathbf{p}_2)}+\overrightarrow{Q}_{u(\mathbf{p}_3)}+\overrightarrow{Q}_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=0\\ \text{with } p_1+p_2=p_3+p_4.\end{aligned} \tag{D.2}$$

where the left-acting supercharges $\overrightarrow{Q}_{\tilde{v}(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{\tilde{v}(\mathbf{p})} = i\left(v_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}+v^*_\alpha(\mathbf{p})\theta\right) \tag{D.3}$$

in contrast to (2.33), because we're acting from the left. It may be easily checked that this indeed produces the correct action of Q on $A^\dagger$. The reader is reminded that the (left-acting) supercharges $\overrightarrow{Q}_{u(\mathbf{p})}$ are defined as

$$\overrightarrow{Q}_{u(\mathbf{p})} = i\left(-u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta}+u^*_\alpha(\mathbf{p})\theta\right). \tag{D.4}$$

Note that

$$\begin{aligned}(\overrightarrow{Q}_{\tilde{v}(\mathbf{p})})^* &= \overrightarrow{Q}_{u(\mathbf{p})}\ ,\\ (\overrightarrow{Q}_{u(\mathbf{p})})^* &= \overrightarrow{Q}_{\tilde{v}(\mathbf{p})}\,.\end{aligned} \tag{D.5}$$

We have used the fact that while complex conjugating, the grassmannian derivatives acting from the left act from the right (and vice-versa) and to bring any such right acting derivative to the left involves introducing an extra minus sign. Armed with the definitions above, we can rewrite (D.2) as (all differential operators henceforth, unless noted otherwise, are taken to act from the left)

$$\left(Q^*_{u(\mathbf{p}_1)}+Q^*_{u(\mathbf{p}_2)}+Q_{u(\mathbf{p}_3)}+Q_{u(\mathbf{p}_4)}\right) S_i(\mathbf{p}_1,\theta_1,\mathbf{p}_2,\theta_2,\mathbf{p}_3,\theta_3,\mathbf{p}_4,\theta_4)=0\,. \tag{D.6}$$

The next step is to observe that

$$\begin{aligned}\left(Q^*_{u(\mathbf{p}_1)}+Q^*_{u(\mathbf{p}_2)}+Q_{u(\mathbf{p}_3)}+Q_{u(\mathbf{p}_4)}\right)\exp(\theta_1\theta_3+\theta_2\theta_4)2p^0_3(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)\\ 2p^0_4(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)=0\end{aligned} \tag{D.7}$$

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after we set $\mathbf{p}_1 = \mathbf{p}_3$ and $\mathbf{p}_2 = \mathbf{p}_4$. We now act on (2.55) with

$$\begin{aligned}\left(Q^*_{u(\mathbf{p}_1)} + Q^*_{u(\mathbf{p}_2)} + Q_{u(\mathbf{p}_3)} + Q_{u(\mathbf{p}_4)}\right) \int d\Gamma \Big[& S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \\ & \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) \\ & S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\Big]. \end{aligned} \tag{D.8}$$

Proceeding with (D.8), one finds

$$\begin{aligned} -\int d\Gamma \Big[& \left(Q_{u(\mathbf{k}_3)} + Q_{u(\mathbf{k}_4)}\right) S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) \\ & 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4) \\ & + S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\ & 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) \left(Q^*_{u(\mathbf{k}_1)} + Q^*_{u(\mathbf{k}_2)}\right) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\Big]. \end{aligned} \tag{D.9}$$

We next integrate by parts keeping in mind that only the derivative parts of the Q change sign (as a consequence of the integration by parts). This gives

$$\begin{aligned} \int d\Gamma \Big[& S_1(\mathbf{p}_1, \theta_1, \mathbf{p}_2, \theta_2, \mathbf{k}_3, \phi_1, \mathbf{k}_4, \phi_2) \\ & \left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\ & 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) S_2(\mathbf{k}_1, \phi_3, \mathbf{k}_2, \phi_4, \mathbf{p}_3, \theta_3, \mathbf{p}_4, \theta_4)\Big]. \end{aligned} \tag{D.10}$$

Here, by $\tilde{Q}_{u(p)}$ and $\tilde{Q}^*_{u(p)}$ we mean

$$\tilde{Q}_{u(p)} = i\left(u_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} + u^*_\alpha(\mathbf{p})\theta\right), \tag{D.11}$$

$$\tilde{Q}^*_{u(p)} = i\left(u^*_\alpha(\mathbf{p})\frac{\overrightarrow{\partial}}{\partial\theta} - u_\alpha(\mathbf{p})\theta\right). \tag{D.12}$$

It can be easily checked (just like (D.7)) that (on setting $\mathbf{k}_3 = \mathbf{k}_1$ and $\mathbf{k}_4 = \mathbf{k}_2$)

$$\begin{aligned} \left(\tilde{Q}_{u(\mathbf{k}_3)} + \tilde{Q}_{u(\mathbf{k}_4)} + \tilde{Q}^*_{u(\mathbf{k}_1)} + \tilde{Q}^*_{u(\mathbf{k}_2)}\right) & \exp(\phi_1\phi_3 + \phi_2\phi_4) 2k_1^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_3 - \mathbf{k}_1) \\ & 2k_2^0 (2\pi)^2 \delta^{(2)}(\mathbf{k}_4 - \mathbf{k}_2) = 0, \end{aligned} \tag{D.13}$$

completing the proof.

E Details of the unitarity equation

In this section, we simplify the unitarity equations (2.63) and (2.64). We define

$$Z(\mathbf{p}_i) = \frac{1}{4m^2} v^*(\mathbf{p}_1) v^*(\mathbf{p}_2)\ v(\mathbf{p}_3) v(\mathbf{p}_4)$$

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and rewrite (2.63) and (2.64) as

$$\begin{aligned}
&\int d\Gamma' \left[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&\quad -Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&\quad +4Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right) \\
&\quad \left.\left.+16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\right] \\
&\qquad = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)
\end{aligned} \tag{E.1}$$

and

$$\begin{aligned}
&Z(\mathbf{p}_i)\int d\Gamma' \left[-4Y(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&+\left(4Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right. \\
&+Y(\mathbf{p}_3,\mathbf{p}_4)\left(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right) \\
&\left.\left.+\frac{1}{4}\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\right)\right]=-2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4).
\end{aligned} \tag{E.2}$$

Since the factor $Z(\mathbf{p}_i)$ depends only on the external momenta $\mathbf{p}_i$, we may evaluate it on the delta functions and this simply yields $Z(\mathbf{p}_i) = 4Y(\mathbf{p}_3,\mathbf{p}_4)$. We finally arrive at

$$\begin{aligned}
&\int d\Gamma' \Bigg[\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\quad -Y(\mathbf{p}_3,\mathbf{p}_4)\Bigg(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\quad +4Y(\mathbf{p}_3,\mathbf{p}_4)\big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\big) \\
&\quad +16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Bigg)\Bigg] \\
&\qquad = 2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4)
\end{aligned} \tag{E.3}$$

and

$$\begin{aligned}
&\int d\Gamma' \Bigg[-16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\quad +Y(\mathbf{p}_3,\mathbf{p}_4)\Bigg(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4) \\
&\quad +4Y(\mathbf{p}_3,\mathbf{p}_4)\big(\mathcal{S}_B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)+\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_B^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\big) \\
&\quad +16Y^2(\mathbf{p}_3,\mathbf{p}_4)\mathcal{S}_F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)\mathcal{S}_F^*(\mathbf{p}_3,\mathbf{p}_4,\mathbf{k}_3,\mathbf{k}_4)\Bigg)\Bigg] \\
&\qquad = -2p_3^0(2\pi)^2\delta^{(2)}(\mathbf{p}_1-\mathbf{p}_3)2p_4^0(2\pi)^2\delta^{(2)}(\mathbf{p}_2-\mathbf{p}_4).
\end{aligned} \tag{E.4}$$

The above equations can be more compactly written as (2.67) and (2.68) respectively (since $p_3\cdot p_4 = p_1\cdot p_2$).

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F Going to supersymmetric Light cone gauge

In this brief appendix we will demonstrate that (upto the usual problem with zero modes) it is always possible to find a super gauge transformation that takes us to the supersymmetric lightcone gauge $\Gamma_- = 0$

Let us start with a gauge configuration that obeys our gauge condition $\Gamma_- = 0$. Starting with this gauge configuration, we will now demonstrate that we can perform a gauge transformation that will take Γ_- to any desired value, say $\tilde{\Gamma}_-$.

Performing the gauge transformation (2.4) we find that the new value of Γ_- is simply $D_- K$. Let

$$K = M + \theta\zeta - \theta^2 P, \tag{F.1}$$

where M, ζ^α, P are gauge parameters. It follows that

$$D_- K = \zeta_- - \theta_-(\partial_{-+}M + P) + \theta_+\partial_{--}M - i\theta_+\theta_-(\partial_{-+}\zeta_- - \partial_{--}\zeta_+) \tag{F.2}$$

Now let us suppose that

$$-\tilde{\Gamma}_- = \chi_- - \theta_-(B + A_{+-}) + \theta_+ A_{--} + i\theta_+\theta_-(2\lambda_- + \partial_{--}\chi_+ - \partial_{-+}\chi_-)$$

We need to find K so that

$$D_- K = \tilde{\Gamma}_-$$

Equating coefficients on the two sides of this equation we find

$$\begin{aligned}
\chi_- + \zeta_- &= 0\,, \\
B + A_{+-} + P + \partial_{-+}M &= 0\,, \\
A_{--} + \partial_{--}M &= 0\,, \\
2\lambda_- + \partial_{--}(\chi_+ + \zeta_+) - \partial_{-+}(\chi_- + \zeta_-) &= 0\,,
\end{aligned} \tag{F.3}$$

which are then solved to get,

$$\begin{aligned}
\zeta_- &= -\chi_-\,, \\
\zeta_+ &= -2\partial_{--}^{-1}\lambda_- - \chi_+\,, \\
M &= -\partial_{--}^{-1}A_{--}\,, \\
P &= -B - A_{+-} + \partial_{-+}(\partial_{--}^{-1}A_{--})\,.
\end{aligned} \tag{F.4}$$

Substituting the above expressions in the expansion for K, we can write

$$K = -\partial_{--}^{-1}A_{--} - i\theta_-(2\partial_{--}^{-1}\lambda_- + \chi_+) + i\theta_+\chi_- + i\theta_+\theta_-(\partial_{-+}\partial_{--}^{-1}A_{--} - B - A_{+-})\,. \tag{F.5}$$

It can be checked that the form of K obtained above follows from

$$K = i\partial_{--}^{-1}D_-\Gamma_-\,, \tag{F.6}$$

which is a supersymmetric version of the gauge transformation used to generate an arbitrary A_- starting from usual lightcone gauge.

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G Details of the self energy computation

In this subsection, we will demonstrate that the self energy $\Sigma(p,\theta_1,\theta_2)$ is a constant independent of the momenta p. As discussed in section 3.3.2 $\Sigma(p,\theta_1,\theta_2)$ obeys the integral equation

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2) \\ - 2\pi\lambda \int \frac{d^3r}{(2\pi)^3} D_-^{\theta_2,-p} D_-^{\theta_1,p}\left(\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}}P(r,\theta_1,\theta_2)\right) \\ + 2\pi\lambda \int \frac{d^3r}{(2\pi)^3}\frac{\delta^2(\theta_1-\theta_2)}{(p-r)_{--}} D_-^{\theta_1,r} D_-^{\theta_2,-r} P(r,\theta_1,\theta_2,)\,. \end{aligned} \tag{G.1}$$

We will now simplify the second and third terms in (G.1). In section 3.3.2 we already observed that the general form of the propagator is of the form given by (3.10). Using the formulae (A.21) and (A.22) we can write (3.10) as

$$P(p,\theta_1,\theta_2) = \left(C_1(p)D^2_{\theta_1,p} + C_2(p)\right)\delta^2(\theta_1-\theta_2) \tag{G.2}$$

In the second term of (G.1) we have to evaluate

$$C_1(p) D_-^{\theta_2,-p} D_-^{\theta_1,p}\left(\delta^2(\theta_1-\theta_2)D^2_{\theta_1,p}\delta^2(\theta_1-\theta_2)\right)\,, \tag{G.3}$$

since the product of $\delta^2(\theta_1-\theta_2)$ vanishes. We further use the formulae (A.22) and then the transfer rule (A.23) to get

$$\begin{aligned} -C_1(p)D_-^{\theta_2,-p}D_-^{\theta_2,-p}\delta^2(\theta_1-\theta_2) &= p_{--}C_1(p)\delta^2(\theta_1-\theta_2) \\ &= p_{--}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,, \end{aligned} \tag{G.4}$$

where we have used the algebra (A.17) in the first line and (A.22) in the second.

Let us now proceed to simplify the third term in (G.1). We need to evaluate

$$\begin{aligned} \delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}\left(C_1(p)D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) + C_2(p)\delta^2(\theta_1-\theta_2)\right) \\ = C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2)\,, \end{aligned} \tag{G.5}$$

where we have used the transfer rule (A.23) and the fact that the product of $\delta^2(\theta_1-\theta_2)$ vanishes. We further simplify

$$\begin{aligned} C_1(p)\delta^2(\theta_1-\theta_2)D_-^{\theta_1,r}D_-^{\theta_2,-r}D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) &= -C_1(p)\delta^2(\theta_1-\theta_2)r_-^{\beta}D_-^{\theta_1,r}D_\beta^{\theta_2,-r}\delta^2(\theta_1-\theta_2) \\ &= C_1(p)\delta^2(\theta_1-\theta_2)r_-^{+}D_-^{\theta_1,r}D_+^{\theta_1,r}\delta^2(\theta_1-\theta_2) \\ &= C_1(p)\delta^2(\theta_1-\theta_2)(-ir_-^{+})D^2_{\theta_1,r}\delta^2(\theta_1-\theta_2) \\ &= r_{--}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,, \end{aligned} \tag{G.6}$$

where in the first line we have used (A.20), in the second line the expression is nonzero for $\beta = -$ and we have used the transfer rule (A.23), while the third line follows from the identity $-iD^2 = D_-D_+$ and the last line follows from the arguments used before.

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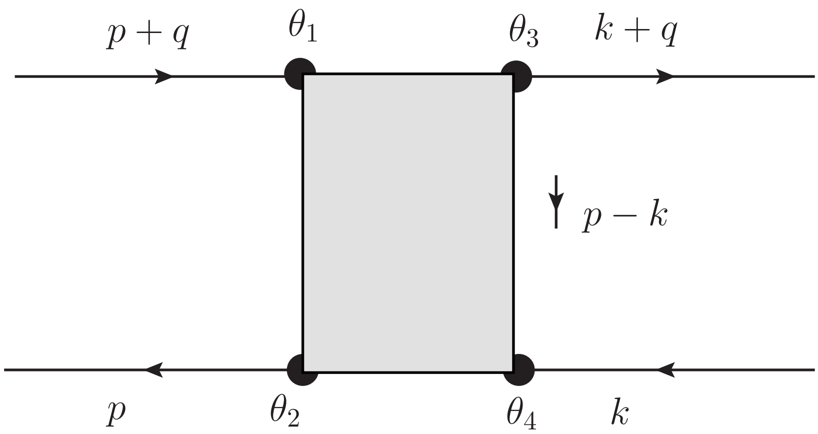

Figure 5. Four point function in superspace.

Thus, using the results (G.6) and (G.4) in (G.1) we get the final form as given in (3.17)

$$\begin{aligned}\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda w \int \frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ -2\pi\lambda\int\frac{d^3r}{(2\pi)^3}\frac{p_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\\ +2\pi\lambda\int\frac{d^3r}{(2\pi)^3}\frac{r_{--}}{(p-r)_{--}}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,. \end{aligned} \tag{G.7}$$

From the above it is clear that the momentum dependence cancels between the second and third terms and we get

$$\Sigma(p,\theta_1,\theta_2) = 2\pi\lambda(w-1)\int\frac{d^3r}{(2\pi)^3}\delta^2(\theta_1-\theta_2)P(r,\theta_1,\theta_2)\,. \tag{G.8}$$

H Details relating to the evaluation of the offshell four point function

H.1 Supersymmetry constraints on the offshell four point function

In this section we will constrain the most general form of the four point function using supersymmetry (see figure 5). Supersymmetric invariance of the four point function in superspace (3.23) implies that

$$(Q_{\theta_1,p+q}+Q_{\theta_2,-p}+Q_{\theta_3,-k-q}+Q_{\theta_4,k})V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)=0\,. \tag{H.1}$$

This can be simplified using (A.24) and written as

$$\sum_{i=1}^{4}\left(\frac{\partial}{\partial\theta_i^\alpha}-p_{\alpha\beta}(\theta_1-\theta_2)^\beta-q_{\alpha\beta}(\theta_1-\theta_3)^\beta-k_{\alpha\beta}(\theta_4-\theta_3)^\beta\right)V(\theta_1,\theta_2,\theta_3,p,q,k)=0\,. \tag{H.2}$$

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We can make the following variable changes to simplify the equation (we suppress spinor indices for simplicity in notation)

$$\begin{aligned} X &= \sum_{i=1}^{4} \theta_i\,, \\ X_{12} &= \theta_1 - \theta_2\,, \\ X_{13} &= \theta_1 - \theta_3\,, \\ X_{43} &= \theta_4 - \theta_3\,. \end{aligned} \tag{H.3}$$

The inverse coordinates are

$$\begin{aligned} \theta_1 &= \frac{1}{4}(X + X_{12} + 2X_{13} - X_{43})\,, \\ \theta_2 &= \frac{1}{4}(X - 3X_{12} + 2X_{13} - X_{43})\,, \\ \theta_3 &= \frac{1}{4}(X + X_{12} - 2X_{13} - X_{43})\,, \\ \theta_4 &= \frac{1}{4}(X + X_{12} - 2X_{13} + 3X_{43})\,. \end{aligned} \tag{H.4}$$

In terms of the new coordinates, the derivatives are then expressed as

$$\begin{aligned} \frac{\partial}{\partial\theta_1} &= \frac{\partial}{\partial X} + \frac{\partial}{\partial X_{12}} + \frac{\partial}{\partial X_{13}}\,, \\ \frac{\partial}{\partial\theta_2} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{12}}, \\ \frac{\partial}{\partial\theta_3} &= \frac{\partial}{\partial X} - \frac{\partial}{\partial X_{13}} - \frac{\partial}{\partial X_{43}}\,, \\ \sum_{i=1}^{4} \frac{\partial}{\partial\theta_i} &= 4\frac{\partial}{\partial X}\,. \end{aligned} \tag{H.5}$$

Using the above, one can rewrite (H.2) as

$$\left(4\frac{\partial}{\partial X} - p.X_{12} - q.X_{13} - k.X_{43}\right) V(X, X_{12}, X_{13}, X_{43}, p, q, k) = 0, \tag{H.6}$$

where $p.X_{12} = p_{\alpha\beta}X_{12}^{\beta}$. The above equation can be thought of as a differential equation in the variables X_{ij} and is solved by

$$V(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k) = \exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right) F(X_{12},X_{13},X_{43},p,q,k). \tag{H.7}$$

This is the most general form of a four point function in superspace that is invariant under supersymmetry.

H.2 Explicitly evaluating V_0

In this subsection, we will compute the tree level diagram for the four point function due to the gauge superfield interaction. (see figure 6). In figure 6 the two diagrams are equivalent

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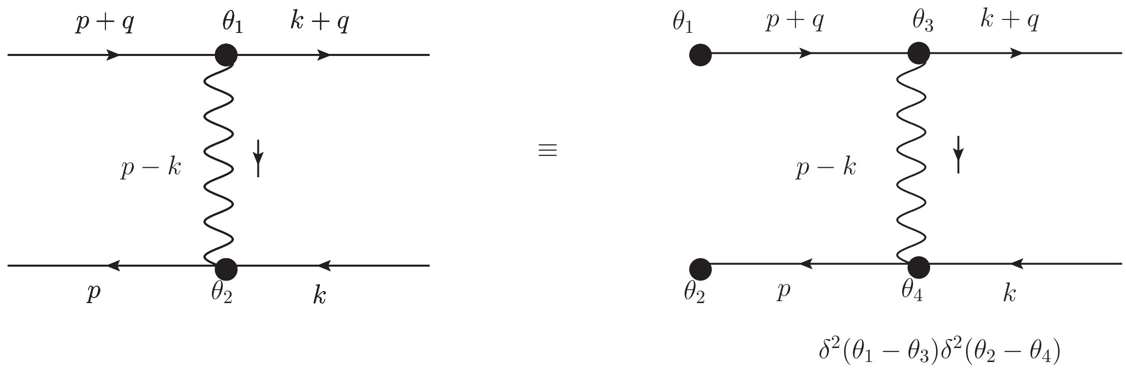

Figure 6. Four point function for gauge interaction: tree diagram.

ways to represent the same process.

$$V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)^{\text{gauge}} = \frac{-2\pi}{\kappa(p-k)_{--}}(D_-^{\theta_2,p}-D_-^{\theta_4,k})(D_-^{\theta_1,p+q}-D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2)\,, \tag{H.8}$$

where $\delta_{ij}^2 = \delta^2(\theta_i-\theta_j)$.[50]

It can be explicitly checked that (see (3.25) for definition of X_{ij})

$$(D_-^{\theta_2,p}-D_-^{\theta_4,k})(D_-^{\theta_1,p+q}-D_-^{\theta_3,-(k+q)})(\delta_{13}^2\delta_{24}^2\delta_{12}^2) = \exp\left(\frac{1}{4}X.(p.X_{12}+q.X_{13}+k.X_{43})\right) F_{\text{tree}}(X_{12},X_{13},X_{43})\,, \tag{H.9}$$

where

$$F_{\text{tree}} = 2iX_{12}^+X_{13}^+X_{43}^+(X_{12}^-+X_{34}^-)\,. \tag{H.10}$$

Thus the final result for the tree level diagram is given by

$$V_0(\theta_1,\theta_2,\theta_3,\theta_4,p,q,k)^{\text{gauge}} = -\frac{4\pi i}{\kappa(p-k)_{--}}\exp\left(\frac{1}{4}X_{1234}.(p.X_{12}+q.X_{13}+k.X_{43})\right) X_{12}^+X_{13}^+X_{43}^+(X_{12}^-+X_{34}^-)\,. \tag{H.11}$$

It is clear that the shift invariant function (H.10) has the general structure of (3.31), with the appropriate identification

$$A(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}}\,,\quad B(p,q,k) = -\frac{4\pi i}{\kappa}\frac{1}{(p-k)_{--}} \tag{H.12}$$

Note that the figure 6 has the $\mathbb{Z}_2$ symmetry (3.28). It is straightforward to check that (H.11) is invariant under (3.28).

H.3 Closure of the ansatz (3.31)

In this section, we establish the consistency of the ansatz (3.31) as a solution of the integral equation (3.29). Consistency is established by plugging the ansatz (3.31) into the r.h.s. of

[50] Note that each vertex factor in figure 6 has a factor of D, resulting in two powers of D in (H.8).

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this integral equation, and verifying that the resultant θ structure is once again of the form given in (3.31). In other words we will show that the dependence of

$$\int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(N V_0(\theta_1, \theta_2, \theta_a, \theta_b, p, q, r) P(r+q, \theta_a, \theta_A) \\ P(r, \theta_B, \theta_b) V(\theta_A, \theta_B, \theta_3, \theta_4, r, q, k) \Big) \tag{H.13}$$

on θ_1, θ_2, θ_3 and θ_4 is given by the form (3.31) with appropriately identified functions A, B, C D.

The algebraic closure described above actually follows from a more general closure property that we now explain. Note that the tree level four point function V_0 (3.30) is itself of the form (3.31). The more general closure property (which we will explain below) is that the expression

$$V_{12} = V_1 \star V_2 \equiv \int \frac{d^3r}{(2\pi)^3} d^2\theta_a d^2\theta_b d^2\theta_A d^2\theta_B \Big(V_1(\theta_1, \theta_2, \theta_a, \theta_b, p, q, r) P(r+q, \theta_a, \theta_A) \\ P(r, \theta_B, \theta_b) V_2(\theta_A, \theta_B, \theta_3, \theta_4, r, q, k) \Big) \tag{H.14}$$

takes the form (3.31) whenever V_1 and V_2 are both also of the form (3.31). In other words (H.14) defines a closed multiplication rule on expressions of the form (3.31).

The explicit verification of the closure described the last paragraph follows from straightforward algebra. Let[51]

$$V_1(\theta_1, \theta_2, \theta_a, \theta_b, p, q, r) = \exp\left(\frac{1}{4} X_{12ab}.(p.X_{12} + q.X_{1a} + r.X_{ba})\right) F_1(X_{12}, X_{1a}, X_{ba}, p, q, r) \tag{H.15}$$

where

$$F_1(X_{12}, X_{1a}, X_{ba}, p, q, r) = X^+_{AB} X^+_{43} \Big(A_1(p,r,q) X^-_{12} X^-_{ba} X^+_{1a} X^-_{1a} + B_1(p,r,q) X^-_{12} X^-_{ba} \\ + C_1(p,r,q) X^-_{12} X^+_{1a} + D_1(p,r,q) X^+_{1a} X^-_{ba} \Big). \tag{H.16}$$

and

$$V_2(\theta_A, \theta_B, \theta_3, \theta_4, r, q, k) = \exp\left(\frac{1}{4} X_{AB34}.(r.X_{AB} + q.X_{A3} + k.X_{43})\right) F_2(X_{AB}, X_{A3}, X_{43}, r, q, k), \tag{H.17}$$

where

$$F_2(X_{AB}, X_{A3}, X_{43}, r, q, k) = X^+_{AB} X^+_{43} \Big(A_2(r,k,q) X^-_{AB} X^-_{43} X^+_{A3} X^-_{A3} + B_2(r,k,q) X^-_{AB} X^-_{43} \\ + C_2(r,k,q) X^-_{AB} X^+_{A3} + D_2(r,k,q) X^+_{A3} X^-_{43} \Big). \tag{H.18}$$

[51] We have used the notations $X_{12ab} = \theta_1 + \theta_2 + \theta_a + \theta_b$ and $X_{AB34} = \theta_A + \theta_B + \theta_3 + \theta_4$.

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Evaluating the integrals over $\theta_a, \theta_b, \theta_A, \theta_B$, we find that V_{12} in (H.14) is of the form (3.31) with

$$
\begin{aligned}
A_{12} &= -\frac{1}{4} q_3 \int d^3\mathcal{R} \Big((C_1 C_2 k_- - D_1 D_2 p_- + 2B_2 C_1 q_3 - 2B_1 D_2 q_3) r_- \\
&\quad + 2A_2 (D_1 p_- + 2B_1 q_3 + 2C_1 r_-) + 2A_1 (C_2 k_- + 2B_2 q_3 + 2D_2 r_-) \Big), \\
B_{12} &= -\frac{1}{4} \int d^3\mathcal{R} \Big((2A_2 - C_2 k_-)(2A_1 + D_1 p_-) + 4B_1 B_2 q_3^2 + 3C_1 D_2 r_-^2 \\
&\quad + (2A_2 C_1 - 2A_1 D_2 - C_1 C_2 k_- - D_1 D_2 p_- + 4B_2 C_1 q_3 + 4B_1 D_2 q_3) r_- \Big), \\
C_{12} &= -\frac{1}{2} \int d^3\mathcal{R} C_2 q_3 (2A_1 + D_1 p_- + 2B_1 q_3 + 3C_1 r_-) \,, \\
D_{12} &= -\frac{1}{2} \int d^3\mathcal{R} D_1 q_3 (-2A_2 + C_2 k_- + 2B_2 q_3 + 3D_2 r_-) \,.
\end{aligned}
\tag{H.19}
$$

where

$$
d^3\mathcal{R} = \frac{d^3 r}{(2\pi)^3} \frac{1}{(r^2 + m^2)((r+q)^2 + m^2)}
$$

It follows from (H.14) that

$$
(V_1 \star V_2) \star V_3 = V_1 \star (V_2 \star V_3)
\tag{H.20}
$$

as both expressions in (H.20) are given by the same integral (the expressions differ only in the order in which the θ and internal momentum integrals are performed). In other words the product defined above is associative. We have directly checked that the explicit multiplication formula (H.19) defines an associative product rule.

H.4 Consistency check of the integral equation

In this section, we demonstrate that the integral equations (3.32)–(3.35) are consistent with the $\mathbb{Z}_2$ symmetry (3.28). First we note that the $\mathbb{Z}_2$ invariance (3.28) of (3.31) imposes the following conditions on the unknown functions of momenta

$$
\begin{aligned}
A(p,k,q) &= A(k,p,-q) \,, \qquad & B(p,k,q) &= B(k,p,-q) \,, \\
C(p,k,q) &= -D(k,p,-q) \,, \qquad & D(p,k,q) &= -C(k,p,-q) \,.
\end{aligned}
\tag{H.21}
$$

These conditions can be written in the form of a matrix given by

$$
E(p,k,q) = T E(k,p,-q) \,,
\tag{H.22}
$$

where

$$
T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \quad E(p,k,q) = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}
\tag{H.23}
$$

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The integral equations (3.32)–(3.35) can be written in differential form by taking derivatives of p_+ and using the formulae in appendix H.5[52]

$$\partial_{p_+} E(p,k,q) = S(p,k,q) + H(p,k_-,q)E(p,k,q) \tag{H.24}$$

where $S(p,k,q)$ is a source term. The equation for k_+ can be obtained from the above equation as follows

$$\begin{aligned}\partial_{k_+} E(p,k,q) &= T\partial_{k_+} E(k,p,-q)\,,\\ &= TS(k,p,-q) + TH(k,p_-,-q)E(k,p,-q)\,,\\ &= TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q)\,,\end{aligned} \tag{H.25}$$

where we have used (H.22). Applying k_+, p_+ derivative on (H.24) and (H.25) respectively and taking the difference we get

$$\begin{aligned}&\partial_{k_+} S(p,k,q) + H(p,k_-,q)\Big(TS(k,p,-q) + TH(k,p_-,-q)TE(p,k,q)\Big)\\ &\quad = T\partial_{p_+} S(k,p,-q) + TH(k,p_-,-q)T\Big(S(p,k,q) + H(p,k_-,-q)E(p,k,q)\Big)\,.\end{aligned} \tag{H.26}$$

Comparing coefficients of $E(p,k,q)$ in the above equation we get the condition

$$[H(p,k_-,q), TH(k,p_-,-q)T] = 0\,. \tag{H.27}$$

For the integral equations (3.32)–(3.35), the $H(p,k_-,q)$ are given by

$$H(p,k_-,q_3) = \frac{1}{a(p_s,q_3)}\begin{pmatrix} (6q_3-4im)p_- & 2q_3(2im+q_3)p_- & (2im+q_3)k_-p_- & -(2im+q_3)p_-^2 \\ 4p_- & 4q_3p_- & -2k_-p_- & 2p_-^2 \\ 0 & 0 & 8q_3p_- & 0 \\ 8im-4q_3 & 4q_3(q_3-2im) & 2(q_3-2im)k_- & (4im+6q_3)p_- \end{pmatrix} \tag{H.28}$$

where

$$a(p_s,q_3) = \frac{\sqrt{m^2+p_s^2}\,\left(4m^2+q_3^2+4p_s^2\right)}{2\pi}\,. \tag{H.29}$$

The matrix $TH(k,p,-q_3)T$ is

$$TH(k,p,-q_3)T = \frac{1}{a(k_s,q_3)}\begin{pmatrix} -(4im+6q_3)k_- & 2q_3(q_3-2im)k_- & -(q_3-2im)k_-^2 & (q_3-2im)k_-p_- \\ 4k_- & -4q_3k_- & -2k_-^2 & 2k_-p_- \\ -8im-4q_3 & 4(-2im-q_3)q_3 & (4im-6q_3)k_- & -(4im+2q_3)p_- \\ 0 & 0 & 0 & -8q_3k_- \end{pmatrix}, \tag{H.30}$$

It is straightforward to check that (H.28) and (H.30) commute. Thus the system of differential equations (H.24) obey the integrability conditions (H.27). Thus the differential equations (H.24) will have solutions that respect the $\mathbb{Z}_2$ symmetry.

[52] Taking derivatives with respect to p_+ eliminates the $r_\pm$ integrals because of the delta functions. The remaining r_3 integrals can be easily performed (see appendix H.5).

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H.5 Useful formulae

The Euclidean measure for the basic integrals are

$$\int \frac{(d^3r)_E}{(2\pi)^3} = \frac{1}{(2\pi)^3} \int r_s dr_s dr_3 d\theta \,, \tag{H.31}$$

where $r_s^2 = r_+ r_- = r_1^2 + r_2^2$ and $r^2 = r_s^2 + r_3^2$. Here the integration limits are $-\infty \le r_3 \le \infty$, $0 \le r_s \le \infty$. Most often we encounter integrals of the type,

$$H(q) = \int \frac{d^3r}{(2\pi)^3} \frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|} \tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{H.32}$$

where we have set $q_\pm = 0$. Another frequently appearing integral is

$$\int \frac{d^3r}{(2\pi)^3} \frac{1}{r^2+m^2} = -\frac{|m|}{4\pi} \tag{H.33}$$

where we have regulated the divergence using dimensional regularization.

In the integral equations (3.32)–(3.35), there are no explicit functions of r_3 appearing in the integral equations and the r_3 integral can be exactly done

$$\int_{-\infty}^{\infty} \frac{dr_3}{(r_s^2+r_3^2+m^2)(r_s^2+(r_3+q_3)^2+m^2)} = \frac{2\pi}{\sqrt{r_s^2+m^2}(4m^2+q_3^2+4r_s^2)} \,. \tag{H.34}$$

The results for the angle integrals are

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(r-p)_- \ (k-r)_-} &= \frac{2\pi}{(k-p)_-} \left(\frac{k_+}{k_s^2} \theta[k_s - r_s] - \frac{p_+}{p_s^2} \theta[p_s - r_s] \right) , \\
\int_0^{2\pi} \frac{d\theta \ r_-}{(r-p)_- \ (k-r)_-} &= \frac{2\pi}{(k-p)_-} \Big(\theta[k_s - r_s] - \theta[p_s - r_s] \Big) , \\
\int_0^{2\pi} \frac{d\theta \ r_-^2}{(r-p)_- \ (k-r)_-} &= -\frac{2\pi}{(k-p)_-} \Big(k_-(1-\theta[k_s-r_s]) - p_-(1-\theta[p_s-r_s]) \Big) .
\end{aligned} \tag{H.35}$$

while the r_s integrals are done with the limits from 0 to ∞. We will also make use of the formula

$$\partial_{\bar z}\left(\frac{1}{z}\right) = 2\pi \delta^2(z, \bar z) \tag{H.36}$$

to derive the differential form of the integral equations.

For doing the angle integrations in (4.10) we used the formula (4.13)

$$\int d\theta \mathrm{Pv} \cot\left(\frac{\theta}{2}\right) \mathrm{Pv} \cot\left(\frac{\alpha-\theta}{2}\right) = 2\pi - 4\pi^2 \delta(\alpha), \tag{H.37}$$

where Pv stands for principal value. This formula is easily verified by calculating the Fourier coefficients as follows

$$\begin{aligned}
\int \frac{d\alpha}{2\pi} e^{-i\alpha} \int d\theta \mathrm{Pv} \cot\left(\frac{\theta}{2}\right) \mathrm{Pv} \cot\left(\frac{\alpha-\theta}{2}\right) &= \oint \frac{d\omega}{2\pi\omega} \omega^{-m} \oint \frac{dz}{z} \mathrm{Pv}\left(\frac{z+1}{z-1}\right) \mathrm{Pv}\left(\frac{z+\omega}{\omega-z}\right) \\
&= \begin{cases} -i \oint dz \, \mathrm{Pv}\left(\frac{z+1}{z-1}\right) z^{-m-1} = -2\pi & (m>0) \\ 0 & (m=0) \\ i \oint dz \, \mathrm{Pv}\left(\frac{z+1}{z-1}\right) z^{-m-1} = -2\pi & (m<0) \end{cases}
\end{aligned} \tag{H.38}$$

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where $z = e^{i\theta}$ and $\omega = e^{i\alpha}$. By comparing (H.38) with Fourier coefficients of delta function,

$$\delta(\alpha) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\alpha}, \tag{H.39}$$

we can immediately check (4.13).

I Properties of the J functions

The J functions are given by

$$\begin{aligned} J_B(q_3, \lambda) &= \frac{4\pi q_3}{\kappa} \frac{n_1 + n_2 + n_3}{d_1 + d_2 + d_3}\,, \\ J_F(q_3, \lambda) &= \frac{4\pi q_3}{\kappa} \frac{-n_1 + n_2 + n_3}{d_1 + d_2 + d_3}\,, \end{aligned} \tag{I.1}$$

where the parameters are

$$\begin{aligned} n_1 &= 16 m q_3 (w+1) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3} + \pi \mathrm{sgn}(q_3)\right)}\,, \\ n_2 &= (w-1)(q_3 + 2im)(2m(w-1) + iq_3(w+3)) \left(-e^{2i\pi\lambda \mathrm{sgn}(q_3)}\right)\,, \\ n_3 &= (w-1)(2m + iq_3)(q_3(w+3) + 2im(w-1)) e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}}\,, \\ d_1 &= (w-1)\left(4m^2(w-1) - 8imq_3 + q_3^2(w+3)\right) e^{4i\lambda\tan^{-1}\frac{2|m|}{q_3}}\,, \\ d_2 &= (w-1)\left(4m^2(w-1) + 8imq_3 + q_3^2(w+3)\right) e^{2i\pi\lambda\mathrm{sgn}(q_3)}\,, \\ d_3 &= -2\left(4m^2(w-1)^2 + q_3^2(w(w+2)+5)\right) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{q_3} + \pi\mathrm{sgn}(q_3)\right)}\,. \end{aligned} \tag{I.2}$$

Both the J functions (I.1) are even functions of q_3

$$J_B(q_3, \lambda) = J_B(-q_3, \lambda)\,,\ J_F(q_3, \lambda) = J_F(-q_3, \lambda)\,. \tag{I.3}$$

Therefore in (I.1) we can replace q_3 with $|q_3|$ and rewrite them as

$$\begin{aligned} J_B(|q_3|, \lambda) &= \frac{4\pi|q_3|}{\kappa} \frac{(\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3)}{(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)}\,, \\ J_F(|q_3|, \lambda) &= \frac{4\pi|q_3|}{\kappa} \frac{(-\tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3)}{(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3)}\,, \end{aligned} \tag{I.4}$$

where

$$\begin{aligned} \tilde{n}_1 &= 16m|q_3|(w+1) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|} + \pi\right)}\,, \\ \tilde{n}_2 &= (w-1)(|q_3| + 2im)(2m(w-1) + i|q_3|(w+3)) \left(-e^{2i\pi\lambda}\right)\,, \\ \tilde{n}_3 &= (w-1)(2m + i|q_3|)(|q_3|(w+3) + 2im(w-1)) e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}}\,, \\ \tilde{d}_1 &= (w-1)\left(4m^2(w-1) - 8im|q_3| + |q_3|^2(w+3)\right) e^{4i\lambda\tan^{-1}\frac{2|m|}{|q_3|}}\,, \\ \tilde{d}_2 &= (w-1)\left(4m^2(w-1) + 8im|q_3| + |q_3|^2(w+3)\right) e^{2i\pi\lambda}\,, \\ \tilde{d}_3 &= -2\left(4m^2(w-1)^2 + |q_3|^2(w(w+2)+5)\right) e^{i\lambda\left(2\tan^{-1}\frac{2|m|}{|q_3|} + \pi\right)}\,. \end{aligned} \tag{I.5}$$

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Another useful way to write the J function is to use the following identities

$$
\begin{aligned}
\tan^{-1}\frac{2m}{q} &= \frac{\pi}{2} - \tan^{-1}\frac{q}{2m} \\
\tan^{-1}\frac{q}{2m} &= \frac{1}{2i}\log\left(\frac{1+\frac{iq}{2m}}{1-\frac{iq}{2m}}\right)
\end{aligned}
\tag{I.6}
$$

Using this relations, it is easy to write the J functions in a factorized form as given in (3.58)

$$
\begin{aligned}
J_B(q,\lambda) &= \frac{4\pi q}{\kappa}\frac{N_1N_2+M_1}{D_1D_2}\,, \\
J_F(q,\lambda) &= \frac{4\pi q}{\kappa}\frac{N_1N_2+M_2}{D_1D_2}\,,
\end{aligned}
\tag{I.7}
$$

where

$$
\begin{aligned}
N_1 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)+(w-1)(2m-iq)\right)\,, \\
N_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(q(w+3)+2im(w-1))+(q(w+3)-2im(w-1))\right)\,, \\
M_1 =& -8mq((w+3)(w-1)-4w)\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\,, \\
M_2 =& -8mq(1+w)^2\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}\,, \\
D_1 =& \left(i\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(w-1)(2m+iq)-2im(w-1)+q(w+3)\right)\,, \\
D_2 =& \left(\left(\frac{2|m|+iq}{2|m|-iq}\right)^{-\lambda}(-q(w+3)-2im(w-1))+(w-1)(q+2im)\right)\,.
\end{aligned}
\tag{I.8}
$$

Another useful property of the J function is manifest in the above form is its reality under complex conjugation

$$
J_B(q,\lambda) = J_B^*(-q,\lambda)\,,\ \ J_F(q,\lambda) = J_F^*(-q,\lambda)\,. \tag{I.9}
$$

Yet another useful way to write the J function is to note that the basic integral which appears in the four point function of scalars in an ungauged theory has the form

$$
H(q) = \int\frac{d^3r}{(2\pi)^3}\frac{1}{(r^2+m^2)((r+q)^2+m^2)} = \frac{1}{4\pi|q_3|}\tan^{-1}\left(\left|\frac{q_3}{2m}\right|\right) \tag{I.10}
$$

for $q_\pm = 0$. Thus we can also write

$$
\begin{aligned}
J_B(|q|,\lambda) &= \frac{4\pi|q|}{\kappa}\frac{N_1N_2+M_1}{D_1D_2}\,, \\
J_F(|q|,\lambda) &= \frac{4\pi|q|}{\kappa}\frac{N_1N_2+M_2}{D_1D_2}\,,
\end{aligned}
\tag{I.11}
$$

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where

$$
\begin{aligned}
N_1 &= \left(e^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)+(w-1)(2m-i|q|)\right) ,\\
N_2 &= \left(e^{-8\pi i\lambda|q|H(q)}(|q|(w+3)+2im(w-1))+(|q|(w+3)-2im(w-1))\right) ,\\
M_1 &= -8m|q|((w+3)(w-1)-4w)e^{-8\pi i\lambda|q|H(q)} ,\\
M_2 &= -8m|q|(1+w)^2e^{-8\pi i\lambda|q|H(q)} ,\\
D_1 &= \left(ie^{-8\pi i\lambda|q|H(q)}(w-1)(2m+i|q|)-2im(w-1)+|q|(w+3)\right) ,\\
D_2 &= \left(e^{-8\pi i\lambda|q|H(q)}(-|q|(w+3)-2im(w-1))+(w-1)(|q|+2im)\right) .
\end{aligned}
\tag{I.12}
$$

I.1 Limits of the J function

I.1.1 $\mathcal{N}=2$ point

The $\mathcal{N}=1$ theory studied in this paper enjoys an enhanced $\mathcal{N}=2$ supersymmetry when $w=1$. Naturally in this limit we expect the J functions to have a simplification. In particular we get

$$
\begin{aligned}
J_B^{w=1} &= -\frac{8\pi m}{\kappa} ,\\
J_F^{w=1} &= \frac{8\pi m}{\kappa} .
\end{aligned}
\tag{I.13}
$$

I.1.2 Massless limit

There exists a consistent massless limit for the J functions

$$
J_B^{m=0}=J_F^{m=0}=\frac{4\pi|q_3|}{\kappa}\frac{(w-1)(w+3)\sin(\pi\lambda)}{(w-1)(w+3)\cos(\pi\lambda)-w(w+2)-5} . \tag{I.14}
$$

This expression is self dual under the duality map (2.12). Note that when $w=1$ this vanishes and is consistent with the $m\to 0$ limit of (I.13).

I.1.3 Non relativistic limit in the singlet channel

The J functions for the S channel are given in (3.76). The non-relativistic limit of the J functions is obtained by taking $\sqrt{s}\to 2m$ with all the other parameters held fixed. In this limit, remarkably we recover the $\mathcal{N}=2$ result.

$$
\begin{aligned}
J_B^{\sqrt{s}\to 2m} &= -\frac{8\pi m}{\kappa} ,\\
J_F^{\sqrt{s}\to 2m} &= \frac{8\pi m}{\kappa} .
\end{aligned}
\tag{I.15}
$$

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